Constitutive Modelling and Flow Simulation of Anisotropic Polar Ice

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Abstract. The different approaches explored by the authors to model the visco-plastic anisotropic behaviour of polar ice associated with the formation and evolution of fabrics, are reviewed. In order to achieve ice rheological models which can significantly improve the simulations of the evolution of ice sheets under varying climatic conditions, these models aim at taking into account the physical mechanisms likely to be active under the conditions prevailing in an ice sheet. Since the destination of a constitutive model for polar ice is its implementation into a large scale ice-sheet model, which is to be run extensively to simulate various climatic scenarios, some compromises must be made to limit its complexity. A possible solution is to use a hierarchy of models of increasing complexity. In this respect the results from the different models are compared and discussed from the viewpoint of ice-sheet flow modelling.

1 Introduction

Ice sheets are an essential ingredient of the climatic system and the first necessary and basic step towards the understanding of how they interact with the other components of the climate, is to be able to describe their response to a given climatic input, that is to solve the ice-sheet flow problem. To this end ice flow modellers need to incorporate in their flow models a rheological model for polar ice, which describes how a representative volume of ice will deform in response to a given stress solicitation. Up to now, all the existing models describe the evolution of an ice sheet by taking into account the variations of the ice-sheet topography and of the boundary conditions which follow directly from the prescribed climate conditions. However, although it is well known now that deformation of polycrystalline ice induces a non random distribution of the orientations of its constituent grains, called fabric or texture, none of these models account for a possible evolution with time of the rheological properties of ice.

According to Duval et al. (1983), under the same prescribed stress, a crystal of ice sheared parallel to its basal planes exhibits a strain rate which can be up to three orders of magnitude greater than when compressed parallel or perpendicular to its c-axis (hexagonal symmetry axis perpendicular to the basal plane). Therefore, since each crystal exhibits a very strong visco-plastic anisotropy, strong fabrics such as observed on ice cores drilled at different sites in Antarctica and Greenland\(^1\) can develop. These macroscopically anisotropic

\(^1\) The observed fabrics are characterized either by crystals' c'-axes alignment along the vertical, as found at Byrd and Law Dome (Antarctica), and at Camp Century and

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structures great Duval, 1982; W 1985; Pimienta (shown by bore-l Shoji and Langy 1988) and by ice Meyssonnier 199 observations evolve quite con face, correspond

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model the visco-plastic and evolution of fabric which can significantly in climatic conditions, seems likely to be active in the evolution of a constitutive sheet model, which is to say that compromises must be made in the evaluation of models in situ models are compared against the first necessity interact with the system to provide a framework for the problem. To this end, rheological models of ice deformation have been developed to study the variations of the fabric evolution. While it is well known now that distribution of the original fabric, none of these models take into account the rows of anisotropic stress strain rates which are consistent or perpendicular to the basal plane of the anisotropic fabric. Stress anisotropy is aligned along the direction of Camp Century and

structures greatly influence the mechanical behaviour of ice (e.g., Bouchez and Duval, 1982; Wilson, 1982; Jacka and Maccagnan, 1984; Azuma and Higashi, 1985; Pimienta et al., 1987), and consequently the flow of ice sheets. This was shown by bore-hole inclinometry surveys (e.g., Russell-Head and Budd, 1979; Shoji and Langway, 1984; Gundes and Hansen, 1984; Hansen and Gundestrup, 1988) and by ice-sheet flow simulations (e.g., Mangeney et al., 1997; Philip and Meyssonnier 1999; Gagliardini and Meyssonnier, 2000). Furthermore, different observations of ice cores show that the grains' crystallographic orientations evolve quite continuously from randomly oriented crystals at the ice-sheet surface, corresponding to macroscopically isotropic ice, to marked preferred orientations with increasing depth, which implies that polar ice experiences microstructural changes during its travel through the ice sheet. Thus the mechanical properties of a representative volume of ice at a given location in the ice sheet may depend on the origin of the ice (e.g., depending on the dust impurity content) but also depend on the strain history that the ice volume has experienced from its deposition at the surface to its present position. As a consequence, a proper simulation of the flow of ice sheets should include a constitutive model which accounts for the strain-induced (evolving) anisotropy of polar ice.

Laboratory studies of texture development in ice, under the conditions existing in polar ice sheets, namely very low deviatoric stresses and temperatures, are not possible because of the too long time required to reach a significant and measurable amount of deformation. Since direct laboratory testing is not feasible and field measurements are not directly usable, the only possibility is to build models. This has been addressed only recently and many models have already been developed to account for the anisotropic behaviour of ice and the evolution of its strain-induced anisotropy, and to simulate the flow of anisotropic ice. Apart from purely phenomenological models (Morland and Staroszczyk, 1998; Staroszczyk and Morland, 2000) which could prove to be very efficient to solve large scale flow of ice sheets, most of them are based on "micro-macro" approaches which allow to derive the macroscopic behaviour of a polycrystal with a given fabric from the assumed known behaviour of the ice grain using a homogenisation procedure (Liboutry, 1993; Azuma 1994; Van der Veen and Whillans, 1994; Castelnau et al., 1996a; Meyssonnier and Philip, 1996; Mangeney et al., 1997 Gödert and Hutter, 1998; Gagliardini and Meyssonnier, 1999a). However, in order to be reliable these models should be based on the physical processes which are known to prevail under in-situ conditions.

The present paper reports the work done by the authors to address the problem of fabric development and of the related mechanical properties of a polycrystal of ice, in order to build a model which could be effectively used as a constitutive model for polar ice in large-scale simulations of ice sheets. First the mechanisms which are likely to be involved in the deformation of polar ice at the grain scale are examined and the consequences on the macroscopic behaviour are

Dye3 (Greenland), or by a random distribution of the c-axes in a vertical plane, as found at Mizuho and Vostok (Antarctica) (Gow and Williamson, 1976; Russell-Head and Budd, 1979; Herron and Langway, 1982; Fujita et al., 1987; Lipk et al., 1989).
discussed. In order to account as much as possible for these physical mechanisms, the different models studied by our group are all based on a micro-macro approach. They differ by the descriptions adopted for the grain behaviour and the fabric, and by the manner the interaction between a grain and its surroundings is accounted for. These different approaches are exposed. Finally the results from the different models are compared and discussed from the viewpoint of ice-sheet flow modelling.

2 Deformation Mechanisms in Polar Ice-Sheets

Owing to the very strong anisotropy of the ice single crystal, a grain in a polycrystal tends to deform mainly by basal glide and its deformation differs from that of the surrounding grains, involving a reorientation of the grain by lattice rotation. Thus the preferred c-axes orientation of polar ice basically develops as the result of lattice rotation by intra-crystalline slip. However, abrupt changes in texture are observed along deep ice cores which are associated with the occurrence of several recrystallization regimes. Therefore, in order to improve the physical basis of polar ice constitutive modelling, the relationship between fabrics and recrystallization regimes needs to be understood.

2.1 Grain Growth and Recrystallization Processes in Ice-Sheets

The evolution of texture in polar ice is achieved via normal grain growth, rotation recrystallization and migration recrystallization (Pimienta and Duval, 1989; Alley, 1992). In the upper layers of ice sheets (several hundred meters), the mean grain size increases with depth. The driving force for normal grain growth, \( F_{gb} = 3 \gamma_{gb} / D \), results from the decrease in free energy linked to the reduction of the grain boundary area. Compared to metals, \( F_{gb} \) is low, less than 100 J m\(^{-3}\), since the grain size \( D \) is generally larger than 1 mm and the grain boundary free energy is \( \gamma_{gb} = 0.065 \) J m\(^{-2}\). This energy is equivalent to a dislocation density of \( 3 \times 10^{11} \) m\(^{-2}\). Grain growth seems to be inhibited by impurities and particles (Alley et al., 1986; Thorsteinsson et al., 1995). The negative correlation between grain-growth rate and impurity concentration observed in the upper part of the GISP2 ice core strengthens the argument that impurities affect grain growth (Alley and Woods, 1996), but does not demonstrate causality. Additional information on the structure of grain boundaries and the physical mechanisms for grain boundary migration in ice should significantly contribute to the determination of the role of impurities in grain growth.

At Byrd, grain growth stops at a depth close to 400 m (Gow and Williamson, 1976). At GRIP, normal grain growth occurs down to 650 m (Thorsteinsson et al., 1997). At Vostok, the first intra-granular sub-boundaries are observed from 700 m (Lipenkov, personal communication). Due to the heterogeneous deformation within the grains, the misorientation of sub-boundaries increases and high angle boundaries develop. This mechanism is called "rotation recrystallization" by the geological community and is also referred to as "continuous recrystallization".

2.2 Deformation

The main feature of shear stresses of the ice by basal glide. The times higher than slip is caused by the vector. But, the ray with this Burgers v. Ahmad and Whittaker faster movement o et al. (1985) to the (1990), long has therefore, cannot n. The creep beha high deviatoric stress more than three or minimum creep ra-ponent \( n \) for secor than 0.2 MPa), the Testa, 1969; Duval ted by field measure.
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d grain growth, rotation and Duval, 1989; hundred meters), the normal grain growth, to the reduction of less than 100 Jm$^{-3}$ grain boundary free a dislocation density purity of impurities and particles correlation between the upper part of the affect grain growth salinity. Additional intrinsic mechanisms for nite to the determin-

2.2 Deformation Mechanisms in Mono and Polycrystalline Ice

The main feature of the plasticity of ice crystals is its outstanding anisotropy. For shear stresses of the order of 0.1 MPa, such as found in active glaciers, ice deforms by basal glide. The resistance to shear on nonbasal planes is large and can be 60 times higher than the resistance on the basal plane (Duval et al., 1983). Basal slip is caused by the glide motion of basal dislocations with $<1120>$ Burgers vector. But, the rapid glide of short edge dislocations on (110) prismatic planes with this Burgers vector was observed by X-ray topography (Higashi et al., 1985; Ahmad and Whitworth, 1988). The fact that basal slip is dominant, in spite of faster movement of dislocations on non-basal planes, is attributed by Higashi et al. (1985) to the large difference in the dislocation density. From Hondoh et al. (1990), long basal screw dislocations are dissociated on the basal plane and, therefore, cannot move on prismatic planes.

The creep behaviour of isotropic polycrystalline ice is well documented. At high deviatoric stresses (higher than 0.2 MPa), the strain rate decreases by more than three orders of magnitude during primary creep (Jacka, 1984). The minimum creep rate is reached after a strain of about 1% and the stress exponent $n$ for secondary creep is close to 3. At low deviatoric stresses (lower than 0.2 MPa), the value of the stress exponent is smaller than 2 (Mellor and Testa, 1969; Duval and Castelnau, 1995). These laboratory results are supported by field measurements (Doake and Wolff, 1985; Dahl-Jensen and Gundestrup
A polynomial flow law with $n=1$ at low stresses was obtained by Lipk et al. (1997) from the densification of bubbly ice. This law was already suggested by Lifshitz (1969) and Hutter (1983) in order to remove the singularity exhibited by Glen's law with $n=3$ at zero stress deviator. Creep experiments were recently carried out on fine-grained ices samples (grain size ranging from 8 to 89 µm) by Goldsby and Kohlstedt (1997). A regime with $n=1.8$ was found at low stresses with a strain-rate dependency on the grain size, implying that grain boundary sliding is the dominant creep mechanism. According to these authors, their flow law can be extrapolated to grain sizes of 1 mm or larger, i.e. for conditions prevailing in ice sheets, and grain boundary sliding could be a dominant mechanism of deformation in large ice sheets (Goldsby and Kohlstedt, 1998). However, this cannot explain the development of fabrics which results from lattice rotation induced by dislocation slip (Azuma and Higashi, 1985; Castelnau et al., 1996b). It clearly appears that Goldsby and Kohlstedt’s (1997) assumption does not hold for polar ice, since intra-crystalline dislocation glide is the predominant deformation mode at high and low stresses. From Pimenta and Duval (1989), the deformation of polar ice at low stresses results from intra-crystalline slip accommodated by grain-boundary migration associated with normal grain growth or rotation recrystallization. A deformation model based on the equilibrium between work hardening and recovery processes has been developed by De

where $b$ is the material constant, the decrease in the notation of the dislocation by De La Chapelle equilibrium is neglected of grain growth. Our model accounts for the lower estimate of the crystal.

3 Constitutions

Basically two theories are developed for the effect of active crystal continuity in the process.

3.1 Crystallography

The classical approach describes the permanent shear strain rate $\tau^s$ acting in (s):

$$\dot{\varepsilon}_s = \tau^s$$

where $\dot{\varepsilon}_s$ is the reference to shear in (s) and $\tau^s$ is a constant.

where $s$ is the de- formation of components $m$ slip plane and the fixed reference in (s) is $L^s_{ij} = \eta^s$.
La Chapelle et al. (1985) and Montagnat and Duval (2000) (migration recrystallization is not considered). The increase in dislocation density $\rho^+$ during the deformation is directly related to the strain rate $\dot{\varepsilon}$ by the Orowan equation

$$\frac{dp^+}{dt} = \dot{\varepsilon} / (bD),$$

where $b$ is the magnitude of the Burgers vector and $D$ the sub-grain or grain size. The decrease in the dislocation density is due to both grain-boundary migration and the formation of grain boundaries by rotation recrystallization. An estimate of the dislocation density along the GRIP and Vostok ice cores was obtained by De La Chapelle et al. (1985). For the two sites, the dislocation density at equilibrium is near $1 \times 10^{11} \text{ m}^{-2}$. This value is compatible with the occurrence of grain growth and rotation recrystallization. On the other hand, this physical model accounts for the deformation of polar ice at low stresses with a stress exponent lower than 2 (Montagnat and Duval, 2000).

3 Constitutive Models for Grain Behaviour

Basically two types of constitutive models have been used. The first type, originally developed for metals, describes crystal visco-plasticity by dislocation glide on active crystallographic planes. The second considers the ice grain as a continuous incompressible medium.

3.1 Crystallographic Slip Planes Model

The classical approach used in crystal visco-plasticity (Kocks et al., 1998) describes the permanent visco-plastic deformation of a crystal by a power law. The shear strain rate $\dot{\gamma}^s$ in the slip system $(s)$ depends on the resolved shear stress $\tau^s$ acting in $(s)$ as

$$\dot{\gamma}^s = \dot{\varepsilon}_0 \left[ \frac{\tau^s}{\tau_c^s} \right]^{\eta - 1} \frac{\tau^s}{\tau_c^s},$$

where $\dot{\varepsilon}_0$ is a reference strain rate, $\tau_c^s$ is a constant which characterizes the resistance to shear in the glide system $(s)$ (and includes the temperature dependency), and $\eta^s$ is a constant. The resolved shear stress on $(s)$ is given by

$$\tau^s = s m^s,$$

where $s$ is the deviatoric stress acting on the crystal, and $m^s$ is the Schmid tensor of components $m^s_{ij} = (b_i n_j + b_j n_i)/2$, $n^s$ and $b^s$ denoting the unit normal to the slip plane and the unit vector in the slip direction (Burgers vector) expressed in a fixed reference frame $\mathcal{R}$, respectively. The velocity gradient resulting from slip in $(s)$ is $\Gamma_{ij} = \dot{\gamma}^s b_i n_j$, and the corresponding strain rate in $\mathcal{R}$ is

$$\dot{\varepsilon}^s = \dot{\gamma}^s m^s.$$
The strain rate \( d \) resulting from slip on several systems is the sum of \( d^s \) over all the active systems:

\[
d = \sum_s d^s .
\]  

(5)

Usually, for the sake of simplicity, the exponent \( n^s \) in (2) is assumed to be the same for all the active slip systems, i.e. \( n^s = n \).

Castelnau et al. (1996a) have adapted this model for ice by considering that the deformation involves the basal, prismatic and pyramidal slip systems. The three glide directions in the basal plane \((0, 0, 0, 1)\) are \(<2, 1, 1, 0>\), the prismatic planes \((0, 1, 1, 0)\) provide three slip systems with the same Burgers vectors as the basal systems, and the pyramidal planes provide six slip systems in the planes \((1, 1, 2, 2)\), with respective Burgers vectors \(<1, 1, 2, 3>\) out of the basal plane.

3.2 Continuous Medium Model

A simpler model, which allows analytical calculations, is obtained by assuming that the ice grain behaves as a continuous incompressible medium. The hexagonal symmetry of ice is taken into account by assuming that the grain is transversely isotropic. The material symmetry reference frame of a grain being denoted by \( R \), with rotational symmetry axis \( x_3 \) along the c-axis, the constitutive relation for linear behaviour is expressed in \( R \) by

\[
\begin{align*}
s_{11} - s_{22} = 2\eta_{12}(d_{11} - d_{22}), & \quad s_{33} = 2\eta_{12} \frac{4\alpha - 1}{3} d_{33} , \\
s_{23} = 2\eta_{12}\beta d_{23}, & \quad s_{31} = 2\eta_{12}\beta d_{31}, \quad s_{12} = 2\eta_{12} d_{12} ,
\end{align*}
\]  

(6)

where \( s \) is the deviatoric stress tensor and \( d \) the strain-rate tensor (e.g., Liboutry, 1993; Meyssonnier and Philip, 1999). In this relation

- \( \eta_{12} \) is the viscosity for shear in the plane of isotropy \((x_1, x_2)\),
- \( \alpha \) is the ratio of the axial viscosity along the \( x_2 \) axis to the axial viscosity\(^2\) in the plane of isotropy \((x_1, x_2)\),
- \( \beta \) is the ratio of the viscosity for shear parallel to the plane of isotropy to \( \eta_{12} \).

When \( \alpha = \beta = 1 \) the medium is isotropic and (6) reduces to the Newtonian viscous law, i.e. \( s_{ij} = 2\eta_{ij} \), with viscosity \( \eta = \eta_{12} \).

A simple generalisation of (6) to non-linear behaviour, similar to that proposed by Johnson (1982) (see Meyssonnier and Philip, 1999), is obtained by replacing \( \eta_{12} \) in (6) by an apparent viscosity defined as

\[
\eta^s = A^{-1/n} \gamma^s (1-n)/n = A^{-1} \tau_0^{1-n} ,
\]  

(7)

\(^2\) The axial viscosity is defined as the halved ratio of the axial deviatoric stress to the axial strain rate in an uni-axial compressive creep test.

3.3 Remarks

Relations (6) and \( R \) attached to the medium (11) reduces to

When the medium

and objective exp:

where \( c \) is the c-axis

an objective expr:

\[
\begin{align*}
\gamma_{ax}^2 &= 3 \text{Tr}(M) , \\
\gamma_{p}^2 &= 4 \text{Tr}(dM) , \\
\tau_{ax}^2 &= 3 \text{Tr}(M) , \\
\tau_{pp}^2 &= \text{Tr}(sMs) .
\end{align*}
\]  

(3)

These quantities

\(^3\) These quantities

\( x_3 \) axis, but they
where, using Lilouhtry’s (1993) notation for the invariants by rotation about the $x_3$ axis
\[
\gamma_{xz}^2 = 3d_{33}^2 , \quad \gamma_1^2 = (d_{11} - d_{22})^2 + 4d_{12}^2 , \quad \gamma_2^2 = 4(d_{23}^2 + d_{31}^2) ,
\]
\[
r_{xz}^2 = 3s_{33}^2 / 4 , \quad r_1^2 = (s_{11} - s_{22})^2 / 4 + s_{12}^2 , \quad r_2^2 = s_{23}^2 + s_{31}^2 ,
\]
$\gamma_0$ and $\tau_0$, are defined by
\[
\gamma_0^2 = \frac{4\alpha - 1}{3} \gamma_{xz}^2 + \gamma_1^2 + \beta \gamma_2^2 , \quad \tau_0^2 = \frac{3}{4\alpha - 1} r_{xz}^2 + r_1^2 + \frac{1}{\beta} r_2^2 .
\]
These two invariants are linked by
\[
\tau_0 = \eta^* \gamma_0 ,
\]
thus
\[
\gamma_0 = A \tau_0^p .
\]
When the medium is isotropic $\alpha = \beta = 1$, $\gamma_0^2 = \gamma^2 = 2d_{ij}d_{ij}$, $r_0^2 = \tau_0^2 = s_{ij}s_{ij}/2$, and (11) reduces to Glen’s law with fluidity parameter $A$.

### 3.3 Remarks

Relations (6) and (8) are valid only when expressed in the local reference frame $\mathcal{R}$ attached to the ice crystal. This does not constitute a major drawback since, by definition, the c-axis orientation remains constant inside a given grain. Thus, when using a micro-macro model, the calculations relative to a grain are performed in the local reference frame $\mathcal{R}$ attached to this grain, and when necessary the results are expressed in the global reference frame $\mathcal{R}$. However, objective frame-indifferent expressions may be obtained by introducing a structure tensor $M$ attached to each crystal and defined as
\[
M = c \otimes c ,
\]
where $c$ is the c-axis unit vector ($c = (0,0,1)$ in $\mathcal{R}$). Using Boehler’s (1975) results, an objective expression of relation (6) is obtained as
\[
s = 2\eta_0s [d + [2(\alpha - \beta)M - \frac{2}{3}(\alpha - 1)I] \text{Tr}(Md) + (\beta - 1)(Md + dMd)] ,
\]
and objective expressions for the invariants (8) are obtained as
\[
\gamma_{xz}^2 = 3 \text{Tr}(Md)^2 , \quad \gamma_1^2 = 2 \text{Tr}(d^2) + 4 \text{Tr}(dMd) , \quad \gamma_2^2 = 4 \text{Tr}(dMd) - \text{Tr}(Md^2) ,
\]
\[
r_{xz}^2 = 3 \text{Tr}(s^2) / 4 , \quad r_1^2 = \text{Tr}(s^2) / 2 + 4 \text{Tr}(Ms)^2 / 4 - \text{Tr}(sMs) , \quad r_2^2 = \text{Tr}(Ms) - \text{Tr}(Ms)^2 .
\]

$^3$ These quantities are invariant by rotation of the local reference frame $\mathcal{R}$ about its $x_3$ axis, but they are not invariant under any change of reference frame.
When the behaviour is linear, the transversely isotropic grain formulation is fully equivalent to the description of crystal visco-plasticity by dislocation glide in the basal, prismatic and pyramidal slip systems (Meyssonnier and Phillip, 2000). By applying relations (2), (3), (4) and (5), and identifying with (6), the relations between the resistances \( r_s^b \) to shear in the basal, prismatic and pyramidal slip systems, denoted by \( r_s^b, r_s^{pr} \) and \( r_s^{py} \), respectively, and the grain model parameters \( \eta_2, \alpha, \beta \), are obtained as

\[
\frac{1}{\eta_2} = \frac{3}{2} \epsilon_0 \left( \frac{1}{r_s^{pr}} + 2 \lambda^2 \frac{1}{r_s^{py}} \right), \quad \alpha = \frac{1}{16 \lambda^2} \frac{r_s^{py}}{r_s^{pr}} + \frac{3}{8}, \quad \beta = \frac{r_s^b}{r_s^{pr}} \frac{r_s^{py}}{r_s^{pr}} + 2 \lambda^2 \frac{r_s^b}{r_s^{pr}}, \tag{15}
\]

where \( c/a \) is the ice lattice parameter, \( \lambda = (c/a)/(1 + (c/a)^2) \) and \( \mu = (1 - (c/a)^2)/(1 + (c/a)^2) \). In the non-linear case this equivalence is no more satisfied. For \( n = 3 \), only the components of \( d \) arising from glide on the basal and prismatic systems are found to exhibit rotational symmetry around the c-axis, which is in agreement with Kamb's (1961) results. However, the transversely isotropic model of grain should be acceptable as far as basal glide remains the dominant mechanism.

4 Modelling the Anisotropic Behaviour of Polar Ice

In addition to a model for the grain behaviour, modelling the macroscopic behaviour of ice requires a description of the structure of the polycrystal. Basically, the models presented assume that there is no correlation between the initial size and shape of the grains and their crystallographic orientations, so that the instantaneous viscoplastic response of a polycrystal depends essentially on the current distribution of the grains orientations.

4.1 Description of the Fabric

Two methods have been used to describe the fabric of a polycrystal. They are closely related to the choice made for the description of the polycrystalline aggregate itself and to the nature of the micro-macro model employed (analytical or numerical).

The first method considers the polycrystal standing for the representative volume of ice, as an aggregate of a finite number of grains. The description of the fabric is given simply by the Euler angles which allow to locate the grain reference frame \( \mathcal{R} \) with respect to the reference frame \( \mathcal{R}^* \) attached to the polycrystal. The number of Euler angles per grain depends on the constitutive model adopted to describe the grain behaviour: for the crystallographic slip planes model, three angles are necessary to give the directions of the c and a axes of the grain; with the transversely isotropic model only two angles are needed to give the c-axis orientation (see Fig. 2a).

The other method considers the polycrystal as made of an infinite number of grains, therefore allowing a continuous description of the fabric by means of an orientation suited for analytic of grain, the ODF orientation \( (\theta, \phi) \), sphere, per unit ar whose orientations

Assuming that the orientation, \( dN_r \), is

\[
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\]

\[
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\]

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\]

4.2 The Differe

Two types of homo, behaviour from the one-site models con eous matrix, the \( i \) macroscopic proper stress model assume to the polycrystal, \( s \) of stress. The basis
grain formulation is by dislocation glide stonier and Philip, tifying with (6), the nematic and pyramidal and the grain model

\[
\frac{\tau_{\xi\eta}}{\tau_{\xi\eta}} + 2\lambda^2 \frac{\tau_{\xi\eta}}{\tau_{\xi\eta}} + 2\mu^2 \tau_{\xi\eta}, \quad (15)
\]

\(\lambda^2\) and \(\mu = (1 - \mu)\) is no more satisfied. basal and prismatic the c-axis, which is transversely isotropic mains the dominant

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macroscopic behav- crystral. Basically, between the initial conditions, so that the is essentially on the
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r the representative he description of the c the grain reference the polycrystal. The re model adopted to planes model, three s of the grain; with d to note the c-axis
an infinite number he fabric by means

of an Orientation Distribution Function (ODF). This method is actually best suited for analytical calculations. When using the transversely isotropic model of grain, the ODF is a function \(f(\theta, \varphi)\) which gives the density of c-axes with orientation \((\theta, \varphi)\), i.e. the relative number of c-axis intersections with the unit sphere, per unit area of the sphere (see Fig. 2b). The relative number of grains whose orientations lie in the interval \((\theta, \theta + d\theta; \varphi, \varphi + d\varphi)\) is then

\[
dN_r = \frac{1}{2\pi} f(\theta, \varphi) \sin \theta d\theta d\varphi. \quad (16)
\]

Assuming that there is no correlation between the volume of a grain and its orientation, \(dN_r\) is also a volume fraction. By definition,

\[
\frac{1}{2\pi} \int_0^{2\pi} \int_0^\pi f(\theta, \varphi) \sin \theta d\theta d\varphi = 1. \quad (17)
\]

For a transversely isotropic polycrystal, \(f\) does not depend on \(\varphi\), i.e. \(f(\theta, \varphi) = f(\theta)\), and for an isotropic polycrystal \(f(\theta, \varphi) \equiv 1\).

**4.2 The Different Micro-Macro Approaches**

Two types of homogenization schemes have been used to model the polycrystal behaviour from the assumed knowledge of the grain behaviour. Self-consistent one-site models consider each crystal as an inclusion embedded in an homogeneous matrix, the "homogeneous equivalent medium" or HEM, whose average macroscopic properties are to be determined. On the other hand, the uniformstress model assumes that the stress in each grain equals the bulk stress applied to the polycrystal, seen as a homogeneous medium experiencing a uniform state of stress. The basis of these homogenization schemes is the local "interaction
formula" which provides a relation between the local stress and strain rate acting on a grain, different from grain to grain, and the corresponding macroscopic quantities.

In the framework of self-consistent one-site models the interaction between a grain and its surroundings is taken into account by considering each grain as an inclusion embedded in the HEM. By convention the quantities related to the HEM are denoted by an overbar symbol, and no special notation is used to refer to quantities related to the inclusion. The interaction formula is obtained, by using Eshelby's (1957) solution for an elastic homogeneous ellipsoidal inclusion in an infinite elastic matrix. This solution, given in terms of displacement inside the inclusion, is transposed in terms of velocity by applying the correspondence principle, with a special treatment for incompressibility (see, Gilormini and Vernusse, 1992; Meyssonnier and Philip, 1996). It is given by

\[ d = [I + S : (\overline{C}^{-1} : C - I)]^{-1} : \overline{d} , \]  

(18)

where \( d \) and \( C \), and \( \overline{d} \) and \( \overline{C} \), are the strain-rate and viscosity tensors in the inclusion and in the HEM, respectively, and \( S \) is the Eshelby tensor. The Eshelby tensor components are given by

\[ S_{ijmn} = \frac{1}{8\pi} (G_{ijkl} + G_{jkli})\overline{C}_{klmn} , \]  

(19)

where the \( G_{ijkl} \) are integrals over the surface of the inclusion, which depend only on the visco-plastic properties of the HEM (up to now unknown) and on the shape of the inclusion. Another form, totally equivalent to (18), is often given as

\[ s - \overline{s} = -C^* : (d - \overline{d}) , \]  

(20)

where \( C^* = C : (S^{-1} - I) \) is the "interaction tensor" (Hill's "constraint tensor").

By comparison, the interaction formula associated with the uniform-stress model is much simpler since it merely expresses the basic hypothesis of the model (no interaction between grains), i.e.

\[ s - \overline{s} = 0 . \]  

(21)

In both models, the macroscopic quantities are defined as the volume average of their microscopic counterparts at the grain scale, e.g.

\[ \overline{d} = <d> , \quad \overline{s} = <s> . \]  

(22)

Assuming that the shape of the grains is ellipsoidal, and that the HEM has a linear behaviour, the strain rate and the stress derived from Eshelby's (1957) solution are uniform inside each grain. For the uniform-stress model this is by definition. To simplify further, the grains are assumed to occupy the same volume.

\[ \text{For the homogeneity of each grain } g \text{ by } g \]  

whereas for the \( s \) through an orientation \( f \) as a weighting orientation have the model of grain is the formula is

\[ \text{When using the obtained as a function constitutive model simply obtained as } \]  

\[ \text{In self-consistent from the consistent deviatoric stress the HEM properties deviatoric stress ex identical shape, th corresponding set the problem is evc Then approximate HEM behaviours, et al., 1996a for details} \]  

5 Modelling

The change in the deformation determined a presented abruptly development is given The rotation is from the transform frame. This relation grain, e.g. the grain and by \( e^g \) in \( R \) at:

\[ \text{The spin tensor is } \]
For the homogenization schemes based on a description of the polycrystal by a finite number of grains \( N \), the averaging formula for a quantity \( x \), defined in each grain \( g \) by \( \tilde{x} \), is then simply

\[
< x > = \frac{1}{N} \sum_{g=1}^{N} \tilde{x}_g,
\]

whereas for the schemes based on a continuous description of the polycrystal through an orientation distribution function \( f \), the averaging formula involves \( f \) as a weighting function. If one assumes that all the grains with the same orientation have the same ellipsoidal shape, and if the transversely isotropic model of grain is used (i.e. only the c-axis direction is involved), the averaging formula is

\[
< x > = \frac{1}{2\pi} \int_{0}^{\pi} \int_{0}^{2\pi} x(\theta, \varphi) f(\theta, \varphi) \sin \theta d\theta d\varphi.
\]

When using the uniform-stress model, the strain rate in each grain is obtained as a function of the macroscopic stress \( \sigma \) by using relation (21) and the constitutive model adopted for the grain, then the macroscopic strain rate is simply obtained as a function of \( \sigma \) from relation (22).

In self-consistent models, the macroscopic mechanical properties are derived from the consistency conditions which express that the macroscopic strain rate and deviatoric stress given by relation (22), in which \( d \) and \( s \) are functions of the HEM properties owing to (18) and (19), are precisely the strain rate and deviatoric stress experienced by the HEM. In the special case of inclusions with identical shape, the two consistency conditions are equivalent. In general the corresponding set of equations is non-linear and has to be solved iteratively. The problem is even more complicated when the grain behaviour is non-linear. Then approximate solutions can be achieved by linearization of the grain and HEM behaviours, using either a secant or tangent approach (see Castelnau et al., 1996a for details).

5 Modelling Fabric Evolution

The change in the crystallographic lattice orientation of the grains during the deformation determines the evolution of the fabric. Since this point is often presented abruptly and without justification in the literature, a more detailed development is given.

The rotation rate of the crystallographic axes of the crystal can be derived from the transformation formula for the spin tensor\(^4\) under a change of reference frame. This relation is obtained by considering how a vector attached to the grain, e.g. the grain c-axis vector expressed by \( \mathbf{c} \) in the fixed reference frame \( \mathcal{K} \) and by \( \mathbf{c}' \) in \( \mathcal{K} \) attached to the grain, transforms under the velocity gradient

\(^4\) The spin tensor is not objective and thus do not transforms as the strain-rate tensor.
denoted by $L$ in $\mathcal{R}$ and by $L^s$ in $\mathcal{R}$. By definition $c$ transforms into $c + dc$, and $c^s$ into $c^s + dc^s$, such that
\[ dc = Lc dt, \quad dc^s = L^s c^s dt. \] (25)

Denoting by $R$ the matrix for the change of reference frame from $\mathcal{R}$ to $\mathcal{R}$, the $c$-axis vector components are linked by
\[ c = Rc^s, \] (26)
which differentiated with respect to time, using (25)$_2$, gives
\[ dc = Rc^s dt + R dc^s = Rc^s dt + R L^s c^s dt. \] (27)

On the other hand, using (26), relation (25)$_1$ can be rewritten as
\[ dc = L Rc^s dt. \] (28)

Denoting by $\omega$ the skew-symmetric spin tensor defined by $d = d + \omega$, and using the transformation formula for the objective tensor $d$, i.e. $d = Rd^s R^{-1}$, where $d^s$ is the expression for $d$ in $\mathcal{R}$, it follows from (27) and (28) that
\[ \left( R^{-1} \omega R - R^{-1} \dot{R} - \omega^s \right) c^s = 0, \] (29)
where $\omega^s$ is the expression for $\omega$ in $\mathcal{R}$. Since this calculation for the transformation of the $c$-axis vector also holds for the two other base vectors of $\mathcal{R}$:
\[ R^{-1} \omega R = R^{-1} \dot{R} + \omega^s. \] (30)

This equation expresses that the spin $\omega$ of the grain decomposes into the sum of the visco-plastic spin $\omega^s$, measured in the rotating reference frame $\mathcal{R}$, plus the rate of rotation of $\mathcal{R}$ with respect to the fixed global frame $\mathcal{R}$. In practice (30) gives the rotation rates $\dot{\theta}$, $\dot{\phi}$ and $\dot{\psi}$ of the Euler angles which define the position of $\mathcal{R}$ with respect to $\mathcal{R}$, as far as $\omega$ and $\omega^s$ are known.

In the framework of self-consistent one-site models, Eshelby's (1957) solution provides the following expression for the spin tensor $\omega$ of the inclusion (see Meyssonier and Philip, 1996)
\[ \omega = \bar{\omega} + \Omega : S^{-1} : (d - \bar{d}), \] (31)
where $\bar{d}$ and $\bar{\omega}$ are the strain rate and the spin experienced by the HEM at infinity, $S$ is the Eshelby tensor given by (19), and $\Omega$ is the skew-symmetric Eshelby tensor defined by
\[ \Omega_{ijmn} = \frac{1}{8\pi} \left( G_{ijkl} - G_{jkil} \right) \bar{C}_{klmn}, \] (32)
where the $G_{ijkl}$ are the same integrals which appear in the expression (19) of $S$. Since the only assumption made in the uniform-stress approach concerns stresses

Constitutive (i.e. $s = \bar{s}$), no kin conditions applied deal with fabric ev experiences the sar

To obtain the relation (30), the c plastic spin measure With the transvers rate of rotation of orientation. Only ti for $\theta$ and $\phi$. They crystal remain para of the velocity alon

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When the ice po tion of the fabric is the rotation of all ti updated from the r appropriate express ent distribution conservation equatic understand by imag spher. Since spont:

5 Since it reminds of t is obviously not co However, it is not le 5 This is a snapshot. spect to $\bar{R}$; since the formula for objectiv
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(i.e. \( s = \bar{s} \)), no kinematic condition exists to infer the spin of a grain from the conditions applied on the HEM boundary. Then, the usual and easiest way to deal with fabric evolution with this type of model, is to assume that each grain experiences the same spin\(^5\) that the HEM, that is

\[
\omega = \bar{\omega} .
\]  

(33)

To obtain the rate of rotation of the crystallographic axes of the grain from relation (30), the other skew-symmetric tensor \(\omega^g\), which represents the viscoplastic spin measured in the rotating reference frame \(\mathcal{R}\), needs to be determined. With the transversely isotropic model of grain, only \(\theta\) and \(\phi\), which define the rate of rotation of the c-axis, need to be known in order to update the grain orientation. Only the components \(\omega^g_{23}\) and \(\omega^g_{31}\) of \(\omega^g\) are necessary to solve (30) for \(\theta\) and \(\phi\). They are obtained by expressing that the basal planes of the ice crystal remain parallel to each other during the deformation (i.e. the component of the velocity along the \(x_3\) axis of \(\mathcal{R}\) is independent of \(x_1\) and \(x_2\), so that

\[
\omega^g_{23} = d^g_{23} , \quad \text{and} \quad \omega^g_{31} = d^g_{31} .
\]  

(34)

Similarly, with the crystallographic slip planes model for the grain behaviour, the fact that the planes of each glide system remain parallel to each other implies that the components of \(\omega^g\) are expressed in the fixed reference frame\(^6\) by

\[
\omega^g_{jk} = \sum_s \left( n^s_i b^s_j - b^s_i n^s_j \right) \gamma^s / 2 ,
\]  

(35)

with the notation defined in Sect. 3.1. The expression to be used in (30) must be expressed in the grain reference frame as\(^6\)

\[
\omega^g = R^{-1} \omega^g\bar{\mathcal{R}} R .
\]  

(36)

When the ice polycrystal is described by a finite number of grains, the evolution of the fabric is straightforward: the fabric is updated by taking into account the rotation of all the constituent grains whose crystallographic orientations are updated from the rotation rates of the Euler angles given by (30), using the appropriate expressions for \(\omega\) and \(\omega^g\). When the fabric is described by an orientation distribution function, the ODF must be updated by solving the ODF conservation equation. For a transversely isotropic grain this relation is easy to understand by imagining the intersections of the grains c-axes with the unit sphere. Since spontaneous generation or disappearance of grains is not allowed

\(^5\) Since it reminds of the Taylor type assumption (uniform strain rate), this assumption is obviously not consistent with the basic assumption of the uniform-stress model. However, it is not less physically based than the uniform-stress assumption itself.

\(^6\) This is a snapshot. Here the local reference frame \(\mathcal{R}\) is assumed to be fixed with respect to \(\bar{\mathcal{R}}\); since there is no rotation of \(\mathcal{R}\) with respect to \(\bar{\mathcal{R}}\), the usual transformation formula for objective tensors applies.
(i.e. fast discontinuous recrystallization is not active), the net flux of grains entering the interval \( \theta, \theta + d\theta; \varphi, \varphi + d\varphi \) equals the increase of the number of grains in the interval, so that, using (16),

\[
\frac{\partial(f \sin \theta)}{\partial t} + \frac{\partial(\dot{\theta} f \sin \theta)}{\partial \theta} + \frac{\partial(\dot{\varphi} f \sin \theta)}{\partial \varphi} = 0.
\]  

(37)

The evolution of the fabric is then given by the solution of (37) in which \( \dot{\theta} \) and \( \dot{\varphi} \) are obtained from (30).

6 Applications to Polycrystal and Ice-Sheet Flow Modelling

Following these lines, three different models for the behaviour of anisotropic ice and the evolution of its anisotropy have been developed\(^7\).

The more sophisticated, denoted by ‘OCPD’ in the following, is the adaptation by O. Castelnau and P. Duval (Castelnau et al., 1996a), of the visco-plastic self-consistent model developed for metals by Leben and Tomé (1993), which is very appropriate for ice owing to the weak non-linearity of the flow law. The ice crystal is assumed to deform by basal, prismatic and pyramidal dislocation glide according to interaction and averaging formulas (20) and (23), respectively. Fabric evolution is calculated from relations (30)-(32) and (35). Since in general the a-axes orientation of the grains is not measured, the a-axes of the modelled polycrystals are assumed to be initially distributed at random. Moreover, the same initial shape (spherical) and size of the grains are assumed, although the grain shape can be allowed to vary during the deformation.

A simpler visco-plastic self-consistent model, named ‘JMAP’, was developed by Meyssonier and Philip (1996) for macroscopically transversely isotropic ice which seems to be the more common feature observed in the ice sheets. Each crystal is considered as a continuous transversely isotropic medium whose behaviour is described by relations (6)-(11), and, rather than considering the polycrystal as an assembly of individual grains, the fabric is described by an ODF \( f(\theta, \varphi) \). The self-consistent scheme is based on the interaction formula (18) and the macroscopic averages (22) are computed using relation (24). The evolution of the fabric is obtained by solving the ODF conservation equation (37), using relations (30)-(32) and (34). Assuming that the HEM is linearly transversely isotropic \( f = f(\theta) \) and that the grains remain spherical during the deformation allows a semi-analytical calculation of the model.

\(^7\) Besides these homogenization schemes, a direct approach is possible by using the finite-element method. However, since it leads to very large sets of equations, it has been restricted until now to two-dimensional problems, namely modelling the behaviour of transversely isotropic S2-columnar ice (see Meyssonier and Philip, 1996; Meyssonier and Philip, 2000).

6.1 Visco-plasticity

The resistances of meters of the OCP to laboratory models are small, but non zero, taking into account the crystals compiled in the prismatic \( \tau_\parallel^{\text{pr}} / \tau_\perp = \tau_\parallel \) and of mechanical test with \( \tau_\parallel^{\text{me}} / \tau_\perp = \tau_\parallel \) (Mansuy et al., 1998). For viscosity of isotropic JMAP model, the behaviour (relations finite-element simulated on a bedded in an isot and, however, the (Mansuy et al., 1 weak resistance to additional simplification of the macroscopic \( v \) JMAP model as

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The third model based on the uniform-stress assumption, denoted by 'OGJM', was developed by Gagliardini and Meyssonnier (Gagliardini, 1999; Gagliardini and Meyssonnier, 1999a) with the objective of implementing it effectively in a numerical code for the simulation of ice-sheet flow. Like in JMAP model, the grain is considered as transversely isotropic and the fabric is described by an ODF. Thus the equations of the model are the same as for JMAP, except for the interaction formulas (18) and (31) which are replaced with (21) and (33) respectively. The relative simplicity of the model allows to derive full analytical solutions for linear behaviour in the more general case of an orthotropic polycrystal, i.e. exhibiting three orthogonal planes of reflective symmetry, under loading conditions which preserve the material symmetries (Gagliardini, 1999; Gagliardini and Meyssonnier, 1999a). In particular, the analytical solutions obtained for the evolution of the ODF lead to a parametrization of the ODF using only three parameters.

6.1 Visco-plastic Behaviour of Anisotropic Ice

The resistances of the slip systems $\tau_c^h$, $\tau_c^{hr}$, $\tau_c^{hp}$ which are the essential parameters of the OCPD model, were determined by comparison of model results to laboratory mechanical tests. In order to reproduce the observed behaviour a small, but non negligible, amount of pyramidal slip needs to be introduced. By taking into account the set of data for isotropic polycrystalline ice and monocrys-
tals compiled by Duval et al. (1983), the optimum ratios of the resistances in the prismatic and pyramidal planes to that in the basal plane are found as $\tau_c^{hr}/\tau_c^h = 20$ and $\tau_c^{hp}/\tau_c^h = 200$ (Castelnau et al., 1996a,b). However, the results of mechanical tests on specimens from the Vostok and GRIP cores are best fitted with $\tau_c^{hr}/\tau_c^h = \tau_c^{hp}/\tau_c^h = 70$ (Castelnau, 1996; Castelnau et al., 1997; Castelnau et al., 1998). For given values of these ratios, $\tau_c^h$ is adjusted to reproduce the viscosity of isotropic ice.

JMAP model parameters are $A$, $\alpha$ and $\beta$ which characterize the grain behaviour (relations 6 and 7). The comparison to Duval et al.'s (1983) data of a finite-element simulation of the deformation of a monocrystalline inclusion em-
bedded in an isotropic ice matrix, indicates that $\alpha$ could be between 1 and 4, and, however, that there is no significant influence of $\alpha$ in the range 0.5 - 10 (Mansuy et al., 1990). Since the essential feature of the grain behaviour is the weak resistance to shear parallel to the basal plane, characterized by $\beta \ll 1$, an additional simplification is made by adopting $\alpha = 1$. In the linear case $n = 1$, the macroscopic viscosity $\eta_{iso}$ of an isotropic polycrystal is then obtained with JMAP model as

$$\eta_{iso} = \eta_2 (1 + \sqrt{1 + 24\beta})/6,$$

(38)

which shows that for small values of $\beta$, the macroscopic viscosity of the isotropic polycrystal is controlled by the grain viscosity for shear in the basal plane $\eta_2$. According to Finiamente et al. (1987), a polycrystal with aligned c-axes deforms about ten times faster than isotropic ice when undergoing simple shear parallel
to the basal plane of its grains. Since in the self-consistent model a grain with a given orientation represents all the grains with the same orientation, Pimienta et al.'s (1987) result can be expressed as $\tilde{\eta}_{\text{iso}} = 10\beta\eta_{12}$, where $\beta\eta_{12}$ is the viscosity for shear parallel to the basal plane of a grain. Taking (38) into account, an estimate of $\beta$ is given by $\beta = 0.04$.

JMAP model parameters derived from relation (15), with $c/a = 1.629$ for the ice lattice parameter, are consistent with those used in OCPD model (assuming a linear behaviour): the values of $\alpha$ and $\beta$ corresponding to $\tau_u^p/\tau_u^b = 20$ and $\tau_u^p/\tau_u^b = 200$ are $\alpha \approx 3.5$ and $\beta \approx 0.05$, and with $\tau_u^p/\tau_u^b = 70$, $\alpha \approx 0.7$ and $\beta \approx 0.02$.

With OGJMJ model, in the linear case, the macroscopic viscosity $\tilde{\eta}_{\text{iso}}$ of an isotropic polycrystal is related to the viscosity for shear parallel to the basal plane of a grain $\beta\eta_{12}$ by

$$\beta\eta_{12} = \frac{2 + 3\beta}{5}\tilde{\eta}_{\text{iso}},$$

(39)

so that the viscosity for shear parallel to the plane of isotropy of a polycrystal with aligned c-axes cannot be less than $0.4 \times \tilde{\eta}_{\text{iso}}$ for the minimum value $\beta = 0$ which corresponds to a grain deforming only by basal glide. Since this value is greater than that expected from Pimienta et al. (1987), the influence of anisotropy on ice-sheet flow described by OGJMJ model is under-estimated. This shows that for the uniform-stress model, $\beta$ should not be considered as an intrinsic parameter for the ice-crystal behaviour, but as a model parameter to be adjusted so as to account for the interactions between grains which cannot be accounted for otherwise.

OCPD model was used to describe the behaviour of polar ice in relation to the symmetries of typical fabrics. Several mechanical tests were performed on GRIP ice samples for different orientations of the specimens with respect to the prescribed stress. The visco-plastic anisotropy was found to increase gradually down to about 2600 m depth, revealing a clear correlation between rheology and fabric, and the experimental responses compared well to that predicted by OCPD model (Castelnau et al., 1997; Castelnau et al., 1998).

Figure 3 shows the evolution of the macroscopic parameters of a linearly transversely isotropic polycrystal undergoing uni-axial compression or tension at a fixed strain rate $\dot{\epsilon}_{33}$, as a function of the equivalent strain defined as $\epsilon_{eq} = \|\dot{\epsilon}_{33}\|t$, computed with JMAP model. The polycrystal instantaneous viscous behaviour is described by a relation similar to (6), $\tilde{\alpha}$ and $\tilde{\beta}$ having the same meaning as $\alpha$ and $\beta$ for the grain. The axial viscosities $\tilde{\eta} = \tilde{\eta}_{33}$ and $\eta = \eta_{33}$ shown in the figure are the viscosities in uni-axial compression along the rotational symmetry axis of the polycrystal and along the grain c-axis, respectively. The more significant variations are for $\tilde{\beta}$ in compression, and for $\tilde{\alpha}$ under tension.

### 6.2 Fabric Development

Figure 4 shows fabric evolution given by JMAP model for a linearly transversely isotropic polycrystal, initially isotropic, undergoing uni-axial compression or ten-
Constitutive Modelling and Flow Simulation of Anisotropic Polar Ice

$\eta$ model a grain with a orientation, Fimienta et al. \(\beta \eta_{12} \) is the viscosity \( \alpha \) into account, an with \( c/a = 1.629 \) for the icPD model (assuming \( \eta \) to \( \tau_e^{12}/\eta^2 = 20 \) and \( \tau_e^{12}/\eta^2 = 70 \), \( \alpha \approx 0.7 \).

\( \tau^\alpha \) of an axial viscosity \( \eta_{iso} \) of an \( x \) parallel to the basal

\( \eta^\alpha \) isotropy of a polycrystal for the minimum value basal glide. Since this \( \Gamma \) (1987), the influence del is under-estimated. not be considered as an model parameter to grains which cannot

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\(\eta = \eta_{iso}\) shown ong the rotational sym-

\(\eta = \eta_{iso}\) respectively. The more

\(\alpha\) a linearly transversely axial compression or ten-
Meyssonier (1999a), assuming that the strain-rate history of ice is described by Dahl-Jensen et al. (1993) one-dimensional flow model. Results very close to each other can be obtained by a proper choice of OGJM anisotropy parameter \( \beta \). Fabric evolution is usually represented by using Thorsteinsson et al.'s (1997) statistical parameter \( R_0 \), which characterizes the strength of the fabric and is defined by

\[
R_0 = 2 \left\| \mathbf{c} \right\| - 1
\]

(40)

where \( \mathbf{c} \) is the c-axis unit vector expressed in a global reference frame, the weighted average is defined by (23) or (24), and the symbol \( \left\| \right\| \) denotes the norm of a vector. \( R_0 \) is 0 for a random fabric, and 1 for a fabric with aligned c-axes. Predictions in close agreement with observations only in the zone where grain growth is involved. However both models are unable to describe correctly the kinetics of fabric evolution with depth. OCPD model as well as OGJM model predict a rate of fabric evolution greater than that observed below 650 m in the GRIP ice core (see Fig. 5a). This discrepancy is attributed by Castaño et al. (1996b) and De La Chapelle et al. (1985) to continuous recrystallization (polygonization), not taken into account by both models. As expected from the analysis of the nucleation and growth of grains during recrystallization, grain growth should not alter the development of fabrics, whereas rotation recrystallization should lead to less pronounced fabrics.

### 6.3 Anisotropic Ice Flow Modelling

OGJM model has been coupled with a finite-element model to simulate the stationary flow of a two-dimensional ice sheet, i.e. plane strain or axi-symmetric flow (Gagliardini, 1999). The finite-element simulation provides the velocity and fabric fields corresponding to a stationary state of the ice sheet (see Fig. 6).

Since OGJM model provides a parametrization of the ODF, the field of fabrics is described by using a quadratic interpolation of the three ODF parameters assigned to each node of the finite-element mesh. For a given field of fabrics and a given field of temperature, the finite-element solution of the flow problem, obtained in terms of velocities, allows to compute the trajectories of the ice particles deposited at the surface, as well as the strain history undergone by ice along these lines. With the assumption of stationary flow, the evolution of the fabric is computed by following each streamline from the surface where the

Dahl-Jensen et al. (1993) model assumes that: 1) the ice deforms under uni-axial compression, with a constant vertical strain rate from the surface down to 1750 m, then a linear decrease with depth down to zero on the bed-rock; 2) the present accumulation rate was the same during the Holocene; 3) the surface and the bed-rock are horizontal, and the ice thickness is a constant.

Since the information contained in \( R_0 \) is relatively poor, the comparison of models results with observations is in fact only on the kinetics of fabric development with depth, i.e. assuming that the models describe correctly the modes of fabric development.

---

**Fig. 5.** Fabric evolution using Dahl-Jens OGJM model for : (a) results of OCPD, (b) results of part (a) model, (c) Orient with respect to the ice flow. The influence of the ODF satisfies the ODF parameter Gagliardini and M asianing to stationary ing for a given fabric convergence is actice.

An application done (Gagliardini, that the modelled 1 000 m down to :
of ice is described results very close to isotropy parameter nsson et al's (1997) of the fabric and is

\[ \mathbf{a} \parallel \mathbf{b} \parallel \mathbf{c} \]
eference frame, the \[ \mathbf{a} \parallel \mathbf{b} \parallel \mathbf{c} \] denotes the fabric with aligned ns only in the zone able to describe cor-
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traces of the story undergone by ow, the evolution of re surface where the forms under uni-axial face down to 1750 m, l-rock; 2) the present surface and the bed-
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\[ \beta = 0, \beta = 0.2, \beta = 0.4, \beta = 0.6 \]

Fig. 5. Fabric evolution versus depth in the GRIP ice core. Data from Thorsteinsson et al. (1997) are shown by circles. (a) Fabric strength parameter \( R_0 \) computed by adopting Dahl-Jensen et al's (1993) ice flow model. The solid lines are obtained with OGJM model for different values of grain anisotropy parameter \( \beta \), the dashed line results from OCPD model (Castelnau et al., 1996b). (b) Parameter \( R_0 \) resulting from the finite-element simulation of stationary flow with OGJM model. For comparison results of part (a) are shown (thin solid line: OGJM, \( \beta = 0.25 \); dashed line: OCPD model). (c) Orientation \( \varphi \) of the polycrystal reference frame (of orthotropic symmetry) with respect to the fixed global frame (see Fig. 6). The shadowed area corresponds to the accuracy of measurements.

ice is assumed to be isotropic. By assuming that the directions of the principal stresses do not change significantly during the time step, the parametrized form of the ODF satisfies the ODF conservation equation (37), while the evolution of the ODF parameters is a function of the deviatoric stresses (Gagliardini, 1999; Gagliardini and Meyssonnier, 1999b). The velocity and fabric fields corresponding to stationary flow are calculated by solving iteratively the velocity problem for a given fabric field, then the fabric problem for a given velocity field, until convergence is achieved.

The influence of both temperature and grain anisotropy has been assessed: prescribing a field of temperature derived from that measured at GRIP slows down the rate of fabric evolution with depth compared to an isothermal flow, and an increase of the grain anisotropy leads to an increase of both velocity and rate of fabric evolution.

An application to the flow of ice from GRIP to GISP2 drilling sites has been done (Gagliardini, 1999; Gagliardini and Meyssonnier, 2000). Figure 5b shows that the modelled fabric strengths reproduce very well the measurements from 1000 m down to 2800 m. Above 1000 m the model predicts a slower increase
Fig. 6. Surface and bedrock topographies between the GRIP and GISP2 boreholes and finite-element computed streamlines. The position of the polycrystal reference frame $\mathcal{R} = \{\overline{z}_1, \overline{z}_2, \overline{z}_3\}$ with respect to the fixed global frame $\{X_1, X_2, X_3\}$ is defined by the angle $\overline{\phi}$.

of the fabric, with computed values of $R_0$ lower than measured (below 2800 m the fabrics induced by dynamic recrystallization cannot be reproduced by the model). Figure 5c shows the deviation $\overline{\phi}$ from the vertical of the $\overline{z}_2$ axis of the polycrystal material symmetry reference frame (see definition of $\overline{\phi}$ in Fig. 6). The modelled deviation is less than 2 degrees. This value is in agreement with the observations since, according to Thorsteinsson et al. (1997), the average of the $c$-axes directions is within 1 to 6 degrees with respect to the axis of the core. As regards the horizontal surface velocities and the magnitude of the accumulation rate, the results of the simulation are in reasonable agreement with field data. Since the finite-element computed streamlines are not vertical (see Fig. 6), the strain rates experienced by the ice, then the accumulated strain calculated in the GRIP bore-hole versus depth, are lower than those given by Dahl-Jensen et al.'s (1993) flow model. As a consequence, adopting this latter as a strain-history input for OGJM and OCPD models leads to too high rates of fabric concentration versus depth (see Fig. 5b). This shows that a model for polycrystal fabric evolution cannot be checked separately against ice-core measured fabrics, the comparison involving necessarily the ice-flow model which determines the strain history of ice.

7 Conclusi

The different approach of a polycrystal fabric, have bee OCPD models available now which can be checked mechanisms act involved is large traction, where each considering the this computational concept, as in JI of the code. OG law for numerical anisotropy.

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7 Conclusion

The different approaches followed by our group to model the constitutive behaviour of a polycrystal of ice and to describe the formation and evolution of its fabric, have been exposed.

OCFD model reaches a high level of complexity, and is one of the best models available now. Since the polycrystal is described by a finite number of grains which can be considered individually, it is easy to account for the physical mechanisms acting at the grain scale level. However, since the number of grains involved is large (more than 200), its implementation in a large scale flow simulation, where each material point is represented by a polycrystal, looks difficult considering the huge amount of data and computer time needed. A remedy to this computational difficulty could be to represent the fabric by using the ODF concept, as in JMAP model: this is not easy as it implies a total reconstruction of the code. OGJM model was deliberately kept simple to serve as constitutive law for numerical simulations of the flow of ice sheets exhibiting strain-induced anisotropy. Its efficiency to determine the velocities and the ice fabrics for complex bed-rock and surface topographies, such as the vicinity of the drilling sites in Greenland located near an ice divide, has been demonstrated. However, since OGJM model does not take into account any interaction between a grain and its surroundings, future work should aim at putting together the best ingredients of our models, i.e. self-consistent scheme associated to fabric description by an ODF and implemented in a finite-element code.

Another significant progress will be to improve the polycrystal model by taking fully into account the physical processes caused by grain to grain interactions. Notably, recrystallization is a major concern, not only because it influences the process of texture development, then the anisotropy of ice, but also because the deformation processes involved at the grain scale, e.g. reflected at the macroscopic level by the value of the stress exponent in the relation between the deviatoric stress and the strain rate, depend on the recrystallisation regimes which are activated. A better knowledge will allow to achieve quantitative criteria to separate the different area corresponding to different visco-plastic regimes in the ice sheet, which is quite ignored at present in current ice-sheet models.

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