Flow-induced anisotropy in polar ice and related ice-sheet flow modelling

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Abstract

As fibers or other crystalline materials exhibiting hexagonal symmetry, the crystal of ice can be orientated by using only one single vector, i.e. its c-axis. Such a characteristic allows to apply specific methods to deal with the properties of the polycrystalline aggregate. Among others, the fabric (texture) of the ice polycrystal can be described by an ODF, i.e. a scalar function of two angles that gives the distribution of the orientation of all the constituents (grains).

This paper presents a strain-induced anisotropic flow law for polycrystalline ice and the associated equations describing the evolution of its fabric. This constitutive law is formulated at the polycrystal scale and tabulated using a micro–macro model. The fabric is defined by the second- and fourth-order orientation tensors for the c-axes, assuming the so-called “invariant-based optimal fitting closure approximation”. Both the anisotropic constitutive law and the fabric evolution equations have been implemented in a finite element code in order to solve large scale ice flow problem. As an application, the flow of an idealized ice sheet over a bumpy bed is studied.

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1. Introduction

In the terrestrial environment ordinary ice crystallizes in the hexagonal system (ice Ih). The viscoplastic deformation of the ice Ih single crystal results essentially from dislocation glide on the basal plane (see Fig. 1a), perpendicular to the hexagonal symmetry axis, called the c-axis [1]. As a consequence, ice Ih is one of the most anisotropic natural materials. For a given equivalent stress, the strain rate for shear parallel to the basal plane is between three and four orders of magnitude larger than under compression perpendicular to the basal plane [2].

Polycrystalline ice at the ice-sheet surface results from the transformation of deposited snow: since the ice crystals are distributed at random, its mechanical behaviour is isotropic. During the gravity driven flow of polar ice, the polycrystal of ice develops a strain-induced fabric (or texture), that is, a preferred orientation of the c-axes of its grains. Observations of deep ice cores drilled in Antarctica and Greenland have shown very different fabric patterns, corresponding to different flow conditions. In the Vostok core (Antarctica), the deep ice exhibits girdle type fabric patterns [3], whereas in the Dome C (Antarctica) and GRIP (Greenland) cores, the c-axes concentrate along the vertical direction to form a fabric with a single maximum [4,5]. Strain-induced fabrics, combined to the strong anisotropy of the single crystal, result in a macroscopic behaviour of polycrystalline ice that is also strongly anisotropic and varies from place to place depending on the flow conditions. As shown experimentally by Pimienta et al. [6], shearing perpendicular to the mean direction of the c-axes of a polycrystal that exhibits a single maximum fabric is about 10 times easier than when shearing an isotropic specimen.

To construct ice-sheet flow models aimed at obtaining accurate information on the origin and the age of ice from deep ice cores, the strong and evolving viscoplastic anisotropy of polar ice must be taken into account. This implies to consider the fabric as an unknown of the ice-sheet flow problem. This paper...
aims at presenting an efficient method that allows to describe the anisotropic behaviour of polycrystalline ice and the evolution of its fabric. The constitutive law, as well as the fabric evolution equations, have been implemented into a 3D finite element code for the flow of ice. The obtained results are both the velocity and the fabric fields. The method presented mostly rely on the fact that, since the ice single-crystal deforms essentially by slip on the basal planes with no preferential glide direction in this plane [7], its hexagonal symmetry axis may be considered as a rotational symmetry axis. Then, each grain of a polycrystal is orientated using only one vector, i.e. its c-axis. As a consequence, the method presented here may present an interest as regards other materials whose constituents exhibit a privileged direction (such as fiber reinforced materials, or minerals of the hexagonal family).

The particularity of dealing with ice-sheet flow modelling is the very large difference in scales between the ice grain ($10^{-3}$ m), the polycrystal ($10^{-1}$ m), and the ice sheet ($10^2$ m). To perform a flow calculation, we then need to develop a method (i) that is efficient with respect to computer time for deriving the polycrystal response for a given fabric and (ii) that uses a few variables to describe the fabric in order to reduce computer memory capacity. The method presented in the present paper is based on a representative polycrystal needs to be stored at each integration point, requires a far too large computer memory capacity. The proposed method fulfils these two criteria: (i) that uses a few variables to describe the fabric in order to reduce computer time for deriving the polycrystal response for a given fabric and (ii) that uses a few variables to describe the fabric in order to reduce computer memory capacity.

2. Model for the anisotropic behaviour of polar ice

The mechanical behaviour of an ice polycrystal is directly linked to the orientation and to the mechanical behaviour of its individual grains. The so-called “micro–macro” models, in which the macroscopic behaviour of a polycrystalline aggregate is derived from the knowledge of the microscopic behaviour of its grains, are widely used to describe the anisotropy of polycrystalline polar ice [8–15]. Most of these models use a discrete description for the fabric (i.e. the polycrystal is described as an assembly of a finite number of grains), so that the implementation of these “discrete” models in a large ice-flow model, where a representative polycrystal needs to be stored at each integration point, requires a far too large computer memory capacity. The method presented in the present paper is based on the results from such micro–macro models, however the fabric is described in a continuum way by using the c-axis orientation tensors, and the equations proposed at the macroscopic scale for the polycrystal are tabulated using the results from a discrete micro–macro model.

2.1. Definitions

Unless stated otherwise, in what follows any non-scalar quantity is expressed in a fixed global Cartesian reference frame [R]. Superscripts $T$ and $D$ denote respectively the transpose of a tensor and its deviatoric part.

The c-axis direction of a grain is located with respect to [R] by two Euler angles: the colatitude $\theta$ and the longitude $\psi$. The unit vector along the c-axis is expressed in [R] as $\mathbf{c} = (\cos\psi \sin\theta, \sin\psi \sin\theta, \cos\theta)^T$ (Fig. 1b). The local reference frame of a grain is denoted by $[R^g]$. The c-axis of the grain is along the $\chi_3$-axis of $[R^g]$; $\mathbf{c}^g = (0, 0, 1)$.

The velocity vector is denoted by $\mathbf{u}$, and the velocity gradient $\mathbf{L} = \nabla \mathbf{u}$ decomposes into its symmetric part, the strain-rate tensor $\mathbf{D}$, and its skewsymmetric part, the rotation rate tensor $\mathbf{W}$, respectively, as

$$\mathbf{L} = \frac{1}{2} (\mathbf{D} + \mathbf{D}^T) \quad \text{and} \quad \mathbf{W} = \frac{1}{2} (\mathbf{D} - \mathbf{D}^T).$$

In the following, the overbar symbol denotes a quantity related to the ice polycrystal or to the ice sheet: for example, $\mathbf{D}$ is the strain-rate tensor experienced by a grain, whereas $\mathbf{D}$ is the strain-rate tensor experienced by the polycrystal.

2.2. Microscale equations

Because the equations proposed at the macroscopic scale (the polycrystal scale), to solve the ice-sheet flow problem, are based on the results from homogenization models, we need a minimum of information on the micro-scale level (i.e. the grain scale). In this section we present briefly the main assumptions made at the grain scale, the constitutive law adopted, and the equation for the rotation rate of its c-axis.

2.2.1. Grain mechanical behaviour

Following Meyssonnier and Philip [12], the ice crystal is assumed to behave as a linearly transversely isotropic viscoplastic medium. The relation between the strain-rate tensor $\mathbf{D}$ and the deviatoric stress tensor $\mathbf{S}$ is expressed as

$$\mathbf{D} = \frac{\nu}{2} \left( \beta \mathbf{S} + 2 \left( \frac{\nu + 2}{\nu - 1} \right) \mathbf{S} \mathbf{D} \right) + (1 - \beta) \left( \beta \mathbf{S} \mathbf{M} + \mathbf{M} \mathbf{S} \mathbf{D} \right),$$

where $\nu$ is the Poisson ratio, and $\beta$ is the viscosity ratio. The deviatoric stress tensor $\mathbf{S}$ is expressed as

$$\mathbf{S} = \frac{1}{2} \left( \mathbf{D} + \mathbf{D}^T \right) \mathbf{D}.$$
where $M = \epsilon \otimes \epsilon$ is a structure tensor, $\psi$ the fluidity for shear parallel to the crystal basal plane, and $\beta$ and $\gamma$ are the two dimensionless parameters that define the grain anisotropy:

- $\beta$ is the ratio of the shear viscosity parallel to the basal plane to that in the basal plane,
- $\gamma$ is the ratio of the viscosity in compression or traction along the $c$-axis to that in the basal plane.

Since the ice crystal deforms mainly by slip on the basal plane, the value of $\beta$ should be significantly less than 1, whereas the value of $\gamma$ is about 1.

### 2.2.2 Rate of rotation of the grain

The preferred $c$-axis orientation of polar ice develops as the result of lattice rotation by intra-crystalline slip as illustrated in Fig. 2. In a large part of the ice sheet, lattice rotation is the major mechanism for fabric development. In the present paper, we assume that it is the sole mechanism involved, so that recrystallization is not accounted for.

The strain $W$ of a grain, relative to the global reference frame $[R]$, decomposes into the sum of the viscousplastic spin $W^f$ measured in the grain reference frame $[R^g]$, plus the rate of rotation of $[R^g]$ with respect to $[R]$.

Following Meyssonnier and Philip [12], we assume that a grain deforms essentially by basal glide so that

$$ W^f_{12} = D^f_{12} \quad \text{and} \quad W^f_{23} = D^f_{23}, $$

(3)

where $D^f$ is the strain rate in $[R^g]$.

With this in hand, the rate of rotation of the $c$-axis can be expressed as

$$ \dot{\psi} = W \cdot \epsilon - D \cdot \epsilon + (\epsilon^T \cdot D \cdot \epsilon) \cdot \epsilon. $$

(4)

This equation is similar to Jeffery’s equation [16] for the motion of a rigid elongated ellipsoid in a Newtonian solvent, widely used to predict fiber orientation in dilute suspensions (it differs only by the signs).

### 2.3 Micro-macro models

To model the flow of polar ice we need a constitutive law that relates the macroscopic strain rate $D$ to the macroscopic deviatoric stress $S$, for a given fabric, and a model for the evolution of the fabric. To this aim we must use a micro-macro model.

The “static model” assumes a uniform state of stress within the polycrystal, which has been widely used mainly because it provides analytical results. According to Castelnau et al. [11], the uniform stress model gives a good estimate of the experimental response of an anisotropic ice polycrystal, but it underestimates the anisotropy of its fabric. The “Taylor model”, that assumes a uniform state of strain-rate within the polycrystal, is not appropriate for ice because of the lack of slip systems. However, it provides an upper bound for the macroscopic behaviour. More physically based models have been developed, among which the viscousplastic self-consistent model (VPSC) [11,12] in which the influence of the neighbourhood of each grain is accounted for by considering this grain as an inclusion embedded in a homogeneous matrix. These intermediate models are generally too complex and time consuming to be implemented directly in an ice flow model.

The equations presented in the following, and used to model the strain-induced fabric development of polar ice, derive from analytical results obtained by using both the static and Taylor models. The free parameters in these equations can be tabulated by using any available micro-macro model, as done in the present paper with the VPSC self-consistent model.

### 2.4 Orthotropic fabric

The ice fabric is described by its second- and fourth-order orientation tensors, denoted by $a^{(2)}$ and $a^{(4)}$, respectively, defined as

$$ a^{(2)} = \int f(\epsilon) \epsilon \otimes \epsilon \, d\epsilon = \langle \epsilon \otimes \epsilon \rangle, $$

$$ a^{(4)} = \int f(\epsilon) \epsilon \otimes \epsilon \otimes \epsilon \otimes \epsilon \, d\epsilon = \langle \epsilon \otimes \epsilon \otimes \epsilon \otimes \epsilon \rangle, $$

(5)

where $f(\epsilon)$ is the orientation distribution function (ODF) of the grain $c$-axes [17]. Because the ODF is normalized:

$$ \int f(\epsilon) \, d\epsilon = 1, $$

(6)

the diagonal components of $a^{(2)}$ are linked by

$$ a^{(2)}_{11} + a^{(2)}_{22} + a^{(2)}_{33} = 1. $$

(7)

To reduce the number of variables, a relation linking $a^{(4)}$ to $a^{(2)}$ is assumed. By construction, owing to this closure approximation, the fabric presents orthotropic symmetries [18].

The adopted closure approximation is the so-called “invariant-based optimal fitting closure approximation” (IBOF)
proposed by Chung and Kwon [19]. Its general form is
\begin{equation}
a^{(1)} = \beta_1 \text{Sym}(I \otimes I) + \beta_2 \text{Sym}(I \otimes a^{(2)}) + \beta_3 \text{Sym}(a^{(2)} \otimes I) + \beta_4 \text{Sym}(a^{(2)} \otimes a^{(2)}) + \beta_5 \text{Sym}(a^{(2)} \otimes a^{(2)} \otimes a^{(2)}) + \beta_6 \text{Sym}(a^{(2)} \otimes a^{(2)} \otimes a^{(2)}). \tag{8}
\end{equation}
where \( I \) is the identity tensor, the operator Sym returns the symmetric part of its argument, and the six functions \( \beta_i \) are functions of the second and third invariants of \( a^{(2)} \), denoted by \( \alpha \) and \( \beta \), respectively. It can be shown that, owing to the symmetries of \( a^{(1)} \) and the normalization condition (7), only three functions \( \beta_i \) are independent. Following Chung and Kwon [19] the three independent functions are taken as complete polynomials of degree 5 in \( \alpha \) and \( \beta \), so that 63 parameters need to be determined. By adding two other relations to insure that the IBOF closure approximation is exact for perfectly aligned fabrics and “girdle” fabrics (transverse isotropy), this number is reduced to 61.

These 61 coefficients have been computed (by least square minimization) so that (8) fits the fourth-order orientation tensor calculated by using a parameterized ODF that has been derived analytically by Gagliardini and Meyssonnier [20] with the uniform stress model (see Appendix A).

2.5. Macroscopic scale equations

2.5.1. Polycrystal behaviour

For the polycrystal, we use the so-called “general orthotropic linear flow law” (GOLF) [21]. The expression that relates the macroscopic deviatoric stress \( \bar{S} \) to the macroscopic strain-rate \( \bar{\varepsilon} \) derives from Boehler’s [22] general expression for a micro–macro model, for instance the VPSC. The micro–macro model is run at each node of a regular ordered grid in the space of the two independent eigenvalues of \( a^{(2)} \) in order to obtain a table of the corresponding \( \beta_i \) at the grid nodes (see [21] for the method). The results are stored in a file, so that this calculation needs to be performed only once, prior to any flow simulation. During a flow simulation, the \( \beta_i \) for a given fabric (i.e. a given second-order orientation tensor) are interpolated through the table values.

2.5.2. Fabric evolution

In order to obtain the evolution of the fabric, the velocity gradient experienced by a grain \( \bar{D} \) and \( \bar{W} \) (in (4)) must be expressed in terms of macroscopic quantities through a homogenization scheme. In general, micro–macro models cannot provide an analytical expression. In order to derive the rate of rotation of the \( c \)-axis, we assume that the product \( \bar{L} \cdot \bar{c} \) is intermediate between that for the uniform stress model (Eq. (4) in which \( \bar{W} = \bar{W} \) and \( \bar{D} \) is given by (2) assuming \( \bar{S} = \bar{S} \)) and that for the uniform strain-rate model (Eq. (4) in which \( \bar{W} = \bar{W} \) and \( \bar{D} = \bar{D} \)). Under these assumptions, the \( c \)-axis rotation rate (4) is rewritten as a function of macroscopic quantities as
\begin{equation}
\dot{c} = \bar{W} \cdot \bar{c} - \bar{C} \cdot \bar{c} + (\bar{c}^T \cdot \bar{C} \cdot \bar{c})/2, \tag{10}
\end{equation}
with, accounting for (2) and (3):
\begin{equation}
\bar{C} = (1 - \alpha)\bar{D}^0 + \alpha \bar{S}/2, \tag{11}
\end{equation}
where \( \alpha \) is a scalar “interaction parameter”. \( \alpha = 0 \) corresponds to the uniform strain-rate assumption, \( \alpha = 1 \) corresponds to the uniform stress model, and in practice \( \alpha \) can be adjusted to fit the results from a self-consistent model (e.g. VPSC).

From Eq. (5), the material derivative of \( a^{(2)} \) is
\begin{equation}
\frac{D a^{(2)}}{Dt} = (\bar{e} \cdot \bar{C}) + (\bar{e} \otimes \bar{C}), \tag{12}
\end{equation}
Using the expression of the grain \( c \)-axis angular velocity (10), the equation for the change of \( a^{(2)} \) is achieved as
\begin{equation}
\frac{D a^{(2)}}{Dt} = \bar{W} \cdot a^{(2)} + \bar{C} \cdot \bar{W} - (\bar{C} \cdot a^{(2)} + a^{(2)} \cdot \bar{C}) + 2\dot{c} \bar{C} \tag{13}
\end{equation}
where \( \bar{C} \) is given by (11). As expected, from the similarity of (4) with the corresponding equation for fibers, Eq. (13) has the same form as that obtained by Advani and Tucker [17].

2.6. Discussion

Because of the normalization condition (7) the fabric is completely described by the five independent components of \( a^{(2)} \).

Note that, contrary to what is currently done in 2D plane-strain flow simulations of fiber suspensions (or fiber reinforced polymers), 2D plane-strain flow of ice does not imply a 2D orientation state. Then, for a polycrystalline material, 2D flow calculations still require three variables \( \{\alpha_{21}, \alpha_{22}, \alpha_{23}\} \) to describe the fabric, instead of two for fibers.

The macroscopic equations presented above have been tabulated with the VPSC self-consistent model and the results (not reproduced here) compared with using directly the VPSC model for ice under tension, compression and simple shear. An important result is that for a given set of grain anisotropy parameters \( (\beta, \gamma) \), a unique choice of the interaction parameter \( \alpha \) in (11) allows to reproduce accurately the fabric evolution predicted by the VPSC model.
3. Ice flow simulation

3.1. Field equations

To focus on the topic of strain-induced evolving anisotropy, we restrict the simulation to that of the 2D flow an isothermal ice sheet. The vector components in the vertical direction are denoted by subscript 3 whereas the horizontal components are denoted by the subscripts 1 and 2 or .

The constitutive law for ice is relation (9). Fabric evolution is obtained by solving Eq. (13) for the evolution of the second-order orientation tensor (3 respectively), and the unknown field is contained implicitly in (18) through the constitutive law (9).

3.2. Numerical method

3.2.1. Stokes equation

The discrete variational form of the set of Eqs. (14) and (15) is obtained by integration, using the test functions , and the unknown field is discretized. The unknown fields and are discretized using the interpolation functions ,

\[ \Psi_i(\mathbf{u}) = \Phi_i(\mathbf{u}) \quad \text{and} \quad \Psi_i(p) = \Phi_i(p), \] (19)

where \( \mathbf{u}' \) and \( p' \) are the values of the velocity and the pressure, respectively, at node \( i \) of the finite element mesh.

The unknown \( \mathbf{u} \) is contained implicitly in (18) through the constitutive law (9).

For a given fabric (i.e., a given \( \alpha^0 \)), the numerical solution of (18) is obtained by the residual free bubbles method [24].

3.2.2. Fabric evolution

To solve the fabric evolution equation we need the Eulerian derivative of \( \alpha^0 \), that is

\[ \frac{\partial \alpha^0}{\partial t} = D_a(2) \frac{\partial \mathbf{u}^0}{\partial t} - \mathbf{u}^0 \cdot \nabla \mathbf{d}^2(2). \] (20)

Introducing the vector notation \( \mathbf{A} = (\alpha^0_1, \alpha^0_2, \alpha^0_3, \alpha^0_2, \alpha^0_1) \), the five equations (20) are rewritten under the form:

\[ \frac{\partial \mathbf{A}_i}{\partial t} + \mathbf{u}^0 \cdot \nabla \mathbf{A}_i + \kappa_i \mathbf{A}_i = f_i \quad (i = 1, \ldots, 5). \] (21)

The expressions for the terms \( \kappa_i \) and \( f_i \) are given in Appendix B.

The five equations are also coupled since each contains the five unknown \( \alpha_i^0 \).

This advection-reaction type set of equations is solved by the discontinuous Galerkin Method [25]. Non-linearities are solved by a Picard type iterative scheme (at step \( n + 1 \), the \( \mathbf{C}_{ij} \) and \( \mathbf{W}_{ij} \) components involved in \( \mathbf{u} \) and \( f \) are evaluated from the solution for the flow at step \( n \), and the components of \( \alpha^0 \) are evaluated from the value of \( \mathbf{d}^2(2) \) at step \( n \).

3.2.3. Free surface equation

The free surface elevation \( E \) is discretized as

\[ E(x, y, t) = \Psi_i(x, y) E^i(t), \] (22)

where \( E^i \) is the value of \( E \) at the \( i \)th node of the discretized ice-sheet surface, and \( \Psi_i \) are the interpolation functions.

The discrete variational form of Eq. (17) is obtained by spatial integration over the ice-sheet surface, using the test function \( \Phi \).

At step \( n + 1 \) the following set of equations is then solved

\[ \frac{\partial E^i}{\partial t} \int_{E_i} \Psi_i \Phi dE_{\text{int}} + E^i \int_{E_i} \mathbf{u}^0 \cdot \nabla \mathbf{A}_i \Phi dE_{\text{int}} + \int_{E_i} (\partial_1 + \partial_2) \Phi dE_{\text{int}} = \int_{E_i} (\partial_3 + \partial_5) \Phi dE_{\text{int}}, \] (23)
where $E_n$ is the ice-sheet surface at step $n$, $V_n = (\partial \cdot /\partial x_1, \partial \cdot /\partial x_2, 0)$ is the horizontal gradient operator and $\bar{u}_n = (\bar{u}_1, \bar{u}_2, 0)$ and $\bar{u}_3$ are taken from the solution of the Stokes problem.

Due to the hyperbolic nature of Eq. (23) the Galerkin method (i.e. $\Phi \equiv \Psi$) does not apply. Stabilization is obtained either by applying the residual free bubble method [24] or the stabilized method [26,27].

At each step, to avoid a distortion of the domain mesh caused by the moving free surface, the nodes of the domain mesh are re-distributed by solving a fictive elasticity problem (the domain is given artificial elastic properties, and it is deformed by prescribing on $E_n$, the displacements that correspond to the movement of the free surface).

4. Application

We restrict our study to that of the steady state of a two-dimensional isothermal ice sheet in the vicinity of an ice divide, under plane strain condition (see Fig. 3).

4.1. Flow-domain definition and boundary conditions

The ice-sheet elevation $E(0)$ at the ice divide ($x_1 = 0$) is about 3 km. The domain studied extends from the ice divide to $x_1 = x_L = 120$ km, that is, about 40 times the ice thickness.

The bedrock is assumed to have a sinusoidal elevation between $x_1 = 0$ and 70 km ($\approx 20 \times$ ice thickness), with a wavelength of 20 km and an amplitude of 400 m, and it is assumed to be flat (elevation 0) for $x_1 > 70$ km. The bedrock equation is then

$$B(x_1) = \begin{cases} 
0.4 \left(1 + \cos \frac{\pi x_1}{10}\right) & \text{for } 0 \text{ km} \leq x_1 \leq 70 \text{ km} \\
0 & \text{for } 70 \text{ km} < x_1 \leq x_L
\end{cases}$$

Actual ice sheets never flow over a flat bedrock: both the wavelength and the amplitude of the bumps are in the range of what can be observed under the Antarctic ice sheet (see for example the bedrock topography around Dome C[28]). The short wavelength disturbances are expected to affect only the basal layers where discontinuous dynamic recrystallization occurs owing to the combination of high stresses and high temperature. The very fast grain boundary migration and the nucleation of new grains destroy the strain-induced fabrics that are replaced with multi-maximum stress-induced fabrics. This warm basal ice is commonly described as isotropic, so that our model is no more relevant here. Moreover, in actual ice sheets the bedrock perturbations are expected to create non-negligible three-dimensional effects. However, this idealized bedrock allows us to discuss the effects of a bump on the bedrock depending on its position with respect to the dome location.

The stress on the ice-sheet free surface is such that

$$\theta \cdot n = \sigma^* n,$$

where $\sigma^*$ is the atmospheric pressure, and $n$ the outward unit normal to the surface.

The accumulation $a$ on the ice-sheet surface is assumed to be a constant.

The temperature at the ice-bedrock interface is assumed to be well below freezing point so that there is neither sliding or melting at the ice-bedrock boundary:

$$\bar{u}(x_1, B(x_1)) = 0.$$  

(26)

The ice divide, located at $x_1 = 0$, is assumed to be a symmetry axis for the 2D flow, thus

$$\bar{u}_1(0, x_3) = 0.$$  

(27)

Following many authors [29–31], an horizontal velocity profile is imposed at the other lateral boundary located at $x_1 = x_L$. As pointed out by these authors, more than 10 times the ice thickness away from the boundary, the flow is insensitive to the detail of the velocity profile at the boundary. In Mangeney et al. [29] and Pettit and Waddington [31] the velocity profile corresponds to that of the laminar flow of an isotropic ice sheet. In the framework of the shallow ice approximation (SIA) [32], Gagliardini and Meyssonnier [30] have derived boundary conditions for the velocities at the lateral boundary of an ice sheet with orthotropic ice. By rewriting the constitutive law (9) under the more condensed form:

$$S = M \cdot D.$$  

(28)

Fig. 3. Schematic of the 2D flow simulation.
in the global reference frame, the derivative of the horizontal velocity profile is achieved as
\[
\frac{\partial \bar{u}}{\partial x_3} = \frac{\rho_g}{M_{133}} \frac{\partial E}{\partial x_3} (x_3 - H),
\]
where
\[
H = H(x_3) = E(x_3) - B(x_3)
\]
is the ice thickness. Integration of (29), taking into account the bedrock boundary condition (26), provides the horizontal velocity as
\[
\bar{u}_1 = \frac{\rho_g}{M_{133}} \int_0^{x_3} \left( \frac{\xi - H(x_3)}{M_{133}} \right) \, dx_3,
\]
and the velocity profile at the lateral boundary \(x_1 = x_2\) may be written as
\[
\bar{u}(x_1, x_3) = A \frac{\xi - H(x_3)}{M_{133}} \, dx_3.
\]

The relation between parameter \(A\) and the prescribed accumulation \(a\) at the ice-sheet surface is found by expressing mass conservation in the steady ice sheet: the integration of the surface accumulation from the divide to the boundary \(x_1 = x_3\) must balance the mass flux through the lateral boundary, that is, with the ice divide at \(x_1 = 0\) and accounting for \(B(x_3) = 0\) and a constant “vertical” accumulation \(a\):
\[
ax_3 = \int_0^{x_3} a(x_1, x_3) \, dx_3.
\]
From (32) and (33):
\[
A = ax_3 \int_0^{x_3} \left( \frac{\xi - H(x_3)}{M_{133}} \right) \, dx_3.
\]

As regards the boundary condition for the evolution of the fabric, the snow deposited at the surface is assumed to transform into isotropic ice, so that the second-order orientation tensor at the surface is
\[
\bar{a}^{(2)} = \frac{1}{2} \mathbf{I} \quad \text{on} \quad x_3 = E(x_3).
\]

4.2. Solution method for steady-state flow

4.2.1. Initial condition

The initial state of the ice sheet is obtained by adopting a simplified ice-sheet geometry and a first guess for the fabric field, then by letting the surface elevation evolve until convergence is achieved.

For the initial shape of the ice sheet, we start from the surface elevation profile given by Vialov [33]:
\[
E(x_3) = E(0)^{\frac{3}{2}} \left( 1 - \left( \frac{x_3}{L} \right)^2 \right)^{\frac{1}{2}},
\]
where \(L\) is the horizontal extent of ice sheet, taken as \(L = 1000 E(0)\). Vialov’s profile is obtained analytically for isotropic ice flowing on a flat bedrock using the shallow ice approximation (SLA) [32].

The fabric field is initialized using a polynomial fit of the data from the GRIP (Greenland) ice core. The initial guess of the fabric is transversely isotropic, with the grain \(c\)-axes gathering around the vertical. It is taken as
\[
a^{(2)}_c = \frac{1}{3} + 2.143 \xi - 5.607 \xi^2 + 7.669 \xi^3 - 3.602 \xi^4,
\]

\[
a^{(2)}_t = \frac{1}{2} \left( 1 - a^{(2)}_c \right),
\]
where \(\xi = (E(x_3) - x_3)/H(x_3)\) is the reduced depth. At the surface \((\xi = 0)\) the ice is isotropic, and at the bottom, along the whole bedrock interface, \(a = 0.936\), which corresponds to a strong peak of \(c\)-axes aligned along the vertical.

The initial geometry of the ice sheet was obtained by solving the flow problem (18) and the equation for the free surface (23) in turn. Convergence was achieved after 250 iterations with a time step \(\Delta t = 4\) years (4 a) followed by 200 iterations with time step \(\Delta t = 2\) a, that is after 1400 years.

4.2.2. Steady-state solution

For the present application we are looking for the stationary solutions for the velocity and fabric fields and for the surface elevation. Eqs. (18), (21) and (23) are solved in turn, each iteratively, using an implicit time integration scheme.

The steady-state solution is sought using a transient numerical scheme, assuming constant boundary conditions except for the lateral velocity profile (32) that is dependent on the surface elevation \(E(x_3)\) and on the fabric through \(M_{133}\), then needs to be updated at each time step.

The adopted algorithm is as follows:

1. (1) at time \(t\) the velocity and pressure fields \((\bar{u}, p)\), the fabric field \((\bar{a}^{(2)})\), and the surface elevation \(E\) are known;
2. (2) the evolution of \(\bar{a}^{(2)}\) during the time step \(\Delta t\) is computed, assuming \(\bar{u}\) and \(E\) fixed (this is an iterative process);
3. (3) the evolution of \(E\), during \(\Delta t\) is computed, assuming \(\bar{u}\) and \(\bar{a}^{(2)}\) fixed;
4. (4) the lateral condition (32) is updated;
5. (5) a solution for \(\bar{u}\) at \(t + \Delta t\) is computed, assuming \(\bar{a}^{(2)}\) and \(E\) fixed;
6. (6) steps (1)-(5) are re-iterated until convergence is achieved for \(\bar{u}, \bar{a}^{(2)}\) and \(E\);
7. (7) time \(t\) is incremented to \(t = t + \Delta t\).

The adopted convergence criterion on a solution vector \(q\) is of the form:
\[
||q_{(n+1)} - q_{(n)}|| < \epsilon ||q_{(n)}||,
\]
for inner and for time iterations with index \((n)\) (i.e. inside the time interval \(\Delta t\), and for the passage from \(t\) to \(t + \Delta t\)). \(||q||\) is the \(L^2\) norm defined as
\[
||q|| = \left( \sum_{i=1}^{N} q_i^2 \right)^{1/2},
\]
where \(q_i\) is the value of \(q\) at node \(i\) and \(N\) is the total number of nodes of the domain mesh. In (38), \(q\) stands for each component of the velocity \(\bar{u}\) in the case of iterations of the Stokes problem,
4.3. Results and discussion

The results of the simulation have been obtained using the set of parameters defined in Table 1.

The regular mesh is composed of 3150 linear quadrilateral elements, with $N_{x_1} = 90$ elements in the horizontal direction and $N_{x_3} = 35$ elements in the vertical direction, that is $(N_{x_1} + 1)(N_{x_3} + 1) = 3276$ nodes. The mesh is more refined near the dome and the bedrock where gradients are larger.

Steady state was achieved after 5500 iterations with time step $\Delta t = 4$ a, i.e. 22,000 a. At the end of the computation the evolution of the free surface was less than 1 mm per year, and the difference in the norm of $\mathbf{a}^{(2)}$ was less than $5 \times 10^{-6}$.

The results of the steady-state simulation (the “steady ice sheet”) are compared to the “initial solution” (the “initial ice sheet”), that is, the solution computed by keeping the fabric fixed as given by Eq. (37) until the free surface equilibrium is achieved.

As shown in Fig. 4a, the free surface of the steady ice sheet is close to that of the initial ice sheet, but the amplitude of the surface undulation is larger. The plots of the surface slopes (Fig 4b) exhibit approximately the same wavelength, however the surface of the steady ice sheet exhibits larger slopes.

Owing to the condition of mass conservation, the two solutions, both for the horizontal and for the vertical velocities at the ice sheet surface, are very close to each other, although the steady-state solution presents larger amplitudes (see Fig. 4c and d).

The fields of the components of the second-order orientation tensor for the steady-state solution exhibit a much stronger spatial variability than that of the initial solution (see Fig. 5).

To illustrate the changes in the fabric, the evolution of the steady-state fabric along two vertical lines at $x_1 = 25$ and 35 km, respectively, has been drawn in Fig. 6. This corresponds to what could be measured in two ice cores drilled from the surface to the downstream side of the second bump on the bedrock, and to the upstream side of the next bump, respectively, as shown on Fig. 3. The three eigenvalues $a^{(2)}_1$, $a^{(2)}_2$ and $a^{(2)}_3$ of the second-order orientation tensor $\mathbf{a}^{(2)}$ is plotted in Fig. 6a and c. The three eigenvalues have been ordered so that $a^{(2)}_1 \leq a^{(2)}_2 \leq a^{(2)}_3$. The angle $\bar{\psi}$ of the first eigenvector $e_1$ of $\mathbf{a}^{(2)}$ (associated to $a^{(2)}_1$ ) with the horizontal basis vector $\bar{e}_1$ of the global reference frame, i.e. the inclination of the (material) orthotropy reference frame with respect to the global reference frame, is shown in Fig. 6b and d. The vertical profiles of the fabric parameters for

![Fig. 4. Results of the 2D flow simulation at the ice-sheet surface: (a) elevation; (b) surface slope; (c) horizontal velocity at the surface; (d) vertical velocity at the surface. Solid lines represent the converged steady-state solution with the 3150-element mesh; dashed lines are for the initial solution (fixed anisotropy) and dotted lines are the steady-state solution with the 1500-element mesh.](image-url)
the steady-state solution, present very different patterns from place to place. In the first “borehole” (at the lee side of the bedrock bump at $x_1 = 25$ km), the fabric strength evolves very slowly in the 500 first meters from the surface, where the ice remains close to isotropic (Fig. 6a). Then the $c$-axes rotate very rapidly to gather along a direction close to the in situ vertical (Fig. 6a and b). Owing to the 2D plane strain condition, the $a_{1}^{(2)}$ and $a_{2}^{(2)}$ eigenvalues at steady state are not equal (contrary to the initial fabric). In the second “borehole”, the fabric strength increases regularly from the surface down to the bedrock, where the $c$-axes are concentrated in a direction inclined of about $\bar{\phi} = 10^\circ$ from the vertical. From the glaciology point of view, this deviation from the vertical highlights the necessity of solving both the ice-flow problem and the fabric evolution problem: it is currently admitted that single maximum fabrics are with a vertical symmetry axis, although a small deviation from the vertical may have considerable effects on the flow.

Finally, the abrupt change in $\bar{\phi}$ in “borehole 1” from the surface down do 500 m deep (Fig. 6b) may look surprising. In fact, the analysis of the steady-state results shows that, at the vertical of the downstream side of the bedrock bump, the upper part of the ice sheet undergoes a horizontal compression, so that the maximum eigenvalue $a_{1}^{(2)}$ is associated to the horizontal direction (the grain $c$-axes tend to rotate toward the direction of compression). As ice goes deeper and deeper, shearing parallel to the flow overall direction increases, the $c$-axes tend to gather along the vertical, and the maximum eigenvalue of $a_{2}^{(2)}$ becomes vertical. This result is certainly not general as it must depend on the ratios of the amplitude and of the wavelength of the bedrock bumps relative to the ice thickness.

To assess the dependence of the results on the discretization in space and in time and on the initialization of the fabric field, the results have been compared to those obtained with a mesh composed of 1500 elements ($N_x = 60$ and $N_z = 25$, i.e. 1586 nodes). The initial fabric field was isotropic inside the whole domain. The steady state was achieved after 5000 iterations with a time step $\Delta t = 8$ a, i.e. 40,000 a. As shown on Figs. 4 and 6,
the results obtained with the coarse and refined meshes are very similar. From our experience, the results are more sensitive to the horizontal discretization ($N_x$) than to the vertical discretization ($N_y$). It is also interesting to note that the convergence is achieved two times faster when the fabric field is initialized with the GRIP fabric profile.

5. Conclusion

A method that allows a complete processing of the evolving strain-induced anisotropy of polar ice has been presented. The two major steps, i.e. the description of the ice fabric and its evolution, and the mechanical behaviour of anisotropic ice, are treated separately at the polycrystal (macroscopic) scale. Their interdependency is expressed by using the second-order orientation tensor for the $c$-axes of the grains of the polycrystal. The ice fabric is assumed to remain orthotropic. It is described by the second- and fourth-order orientation tensors for the $c$-axes. Fabric evolution is assumed to be controlled essentially by the rotation of the grain lattice that results from the deformation. An equation for the evolution of the second-order orientation tensor has been derived by using analytical results obtained with the uniform stress and uniform strain-rate homogenization schemes, with the introduction of the interaction parameters (e.g. $k_1$ and $k_2$) and the three Euler angles that locate the orthotropic reference frame with respect to the global reference frame. The case $k_1 = k_2 = k_3 = 1$ represents an isotropic fabric (three dimensional, perfectly random), and $k_1 = k_2 < k_3$ represents a planar random fabric in the plane $(1, 2)$ and $k_3 < 1$ with $k_2 = k_1$ represents a perfectly aligned fabric along direction $1$. Staroszczyk and Gagliardini [15] have shown that this particular parameterized ODF is a good compromise to represent the fabric patterns observed in ice sheets. Because using the second-order orientation tensor assuming a closure approximation for the fourth-order orientation tensor or the parameterized ODF (A.1) to describe the fabric leads to the same kind of orthotropy, we chose the fit the free coefficients in the IBOF closure approximation (8) by comparison with the parameterized ODF.

In the plane of the two independent ODF parameters ($k_1$, $k_2$), the triangular domain defined by $0.02 \leq k_1 \leq k_2 \leq k_3$ is discretized with a regularly spaced grid. At each of the $N = 813$ grid nodes, the second- and fourth-order orientation tensors $a_{ij}^{(2)}$ and $a_{ij}^{(4)}$, respectively are calculated from Eqs. (41) and (51). The fourth-order orientation tensor derived from the IBOF closure approximation, $a_{ij}^{(4)}$, is calculated using (8) in which $a_{ij}^{(2)} = a_{ij}^{(2)}_{\text{IBOF}}$. The 61 independent polynomial coefficients for the IBOF closure approximation are thus calculated by minimizing:

$$x^2 = \sum_{i=1}^{N} (a_{ij}^{(4)} - a_{ij}^{(4)}_{\text{IBOF}})^2 = (a_{ij}^{(4)} - a_{ij}^{(4)}_{\text{IBOF}}),$$

(A.2)

over the $N = 813$ grid nodes. This calculation is made once and for all and depends neither on the crystal behaviour nor on the micro–macro model used to derive the polycrystal properties.

Appendix B. Expression of the coefficients for the evolutions of $a_{ij}^{(2)}$ (Eq. 21)

$a_{ij}^{(4)}$ is calculated as a function of $a_{ij}^{(2)}$ using the IBOF closure approximation.

Only nine coefficients are independent [19]. If one chooses $a_{1111}, a_{2222}, a_{1122}, a_{1212}, a_{2212}, a_{1133}, a_{1112}, a_{2222}$ and $a_{2222}$, then the other components can be deduced from
\[ a_{ij} = a_{ij} - \bar{a}_{ij} - a_{ij,\text{strain}} \quad (i, j, k, l, m = 1, 2, 3; k \neq l \neq m). \]  

(B.1)

The terms \( \tau_i \) in Eq. (21) are then given by

\[ \begin{align*}
\kappa_1 &= 2(\bar{C}_{11} - \bar{C}_{22}), \\
\kappa_2 &= 2(\bar{C}_{22} - \bar{C}_{33}), \\
\kappa_5 &= -3\bar{C}_{33}, \\
\kappa_4 &= \bar{C}_{22} - \bar{C}_{33}, \\
\kappa_3 &= \bar{C}_{11} - \bar{C}_{33},
\end{align*} \]

(B.2)

where \( \bar{C}_{ij} \) is given by (11), and the components of \( f \) are

\[ \begin{align*}
f_1 &= 2(\bar{W}_{12}a_{12} - \bar{W}_{13}a_{13}) + 2(\bar{C}_{12}(2a_{111} - a_{12}) \\
&+ \bar{C}_{13}(2a_{111} - a_{13}) + a_{111}C_{11} + a_{112}C_{22} \\
&- (a_{111} + a_{112})C_{33} + 2a_{111}C_{23}), \\
f_2 &= 2(-\bar{W}_{12}a_{12} - \bar{W}_{13}a_{13}) + 2(\bar{C}_{12}(2a_{222} - a_{12}) \\
&+ \bar{C}_{13}(2a_{222} - a_{13}) + a_{221}C_{11} + a_{222}C_{22} \\
&- (a_{221} + a_{222})C_{33} + 2a_{222}C_{33}), \\
f_3 &= \bar{W}_{12}(a_{12} - \bar{a}_{12}) + \bar{W}_{13}(a_{13} - \bar{a}_{13}) \\
&+ \bar{C}_{13}(4a_{1222} - (a_{12} + a_{13}) + \bar{C}_{13}(4a_{1222} - a_{23}) \\
&+ \bar{C}_{23}(4a_{2222} - a_{13} + 2\bar{a}_{211}C_{11} + a_{222}C_{22} - (a_{211} + a_{222})C_{33}), \\
f_4 &= \bar{W}_{12}(a_{12} - \bar{a}_{12}) + \bar{W}_{13}(a_{13} - \bar{a}_{13}) \\
&+ \bar{C}_{13}(4a_{1222} - a_{23}) + \bar{C}_{23}(3a_{222} - 4a_{2222} + 4\bar{a}_{211} \\
&+ 2\bar{a}_{222}C_{11} + a_{222}C_{22} - (a_{211} + a_{222})C_{33}), \\
f_5 &= \bar{W}_{12}(a_{12} - \bar{a}_{12}) + \bar{W}_{13}(a_{13} - \bar{a}_{13}) \\
&+ \bar{C}_{13}(4a_{1222} - a_{23}) + \bar{C}_{23}(3a_{222} - 4a_{2222} + 4\bar{a}_{211} \\
&+ 2\bar{a}_{222}C_{11} + a_{222}C_{22} - (a_{211} + a_{222})C_{33}).
\end{align*} \]

(B.3)

References


