Plane Flow of an Ice Sheet Exhibiting Strain-Induced Anisotropy

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Abstract. A model for the anisotropic behaviour of polar ice and the evolution of its strain-induced anisotropy is presented. At the scale of the ice polycrystal, the ice fabric is described by a continuous Orientation Distribution Function (ODF), and the stress in each grain is assumed to be the same as the bulk stress (static model). Assuming a linear transversely isotropic behaviour of the ice single crystal, the constitutive law for an orthotropic polycrystal is obtained, as well as the analytical expression for the ODF which depends on three independent parameters only. Applications to the large-scale flow of an ideal ice-sheet are presented. Assuming a fixed geometry of the ice-sheet, the velocities and the fabrics corresponding to stationary plane-strain flow are obtained by solving a coupled problem.

1 Introduction

Observations on ice cores drilled in Antarctica and Greenland have shown that polycrystalline ice develops a fabric (i.e. a lattice preferred orientation of its constituent grains), which is induced by the strain undergone by the ice as it descends from the free surface to the depth of an ice-sheet.

Since it is not possible to reproduce experimentally the conditions which occur in an ice-sheet, the results from models for the anisotropic behaviour of ice must be compared with field data, as has been done, for example, in [2]. The difficulty is that these models require, as input, the knowledge of the strain history undergone by the ice. Results from anisotropic ice flow models which use the ice fabric as an input (e.g.[3]) can be useful, but, since fabrics can be measured only at a very few drilling sites, additional assumptions must be made to extrapolate these fabrics to the whole ice-sheet flow domain. This shows that in order to properly account for the anisotropy of ice we have to consider the ice fabric as an unknown of the ice-sheet flow problem.

On the other hand, a common feature of ice-sheet models is the very large amount of data which are processed. Even by discretizing the flow domain with a coarse mesh (e.g. 30km in the horizontal plane, 30m along the vertical, without a detailed description of the bed-rock topography) some $10^9$ to $10^6$ nodes are required to solve the isotropic ice flow problem. In order to make an ice-sheet model usable when extending it to the flow of anisotropic ice, the models for the behaviour of anisotropic ice and fabric evolution must be as efficient as possible and not too much time consuming.

In this paper, a complete solution (i.e., velocity and fabric fields) for the stationary flow of a two-dimensional ice-sheet is presented. This solution was obtained by incorporating a micro-macro model ([4,5]) for the behaviour of anisotropic ice into a finite-element code for the ice flow simulation. The micro-macro model considers a polycrystal of ice as a representative elementary volume of ice which stands as a material point of the continuum at the scale of the ice-sheet. Some of the assumptions of this model, presented in the following, seem to be rather crude but their aim is to fulfill the main objective of minimizing the amount of data storage and computation time.

2 Rheological model for orthotropic polycrystalline ice

2.1 Notation and main assumptions

In the following three Cartesian reference frames are used (see Fig. 1):

- \{\mathcal{R}\} is a global fixed reference frame, whose plane \((x_1, x_2)\) is the plane of the ice-sheet flow.
- \{^o\mathcal{R}\} is the material reference frame of an orthotropic polycrystal of ice, whose axes \(^o x_1\) are perpendicular to the planes of orthotropic symmetry. Furthermore, it is assumed that the plane of symmetry \((^o x_1, ^o x_2)\) of the polycrystal coincides with the plane \((x_1, x_2)\) of the ice-sheet flow (i.e. \(^o x_3 = x_3\)).
- \{^g\mathcal{R}\} is a local frame attached to an individual grain, whose \(^g x_3\)-axis is the hexagonal symmetry axis of the grain (\(c\)-axis).

![Diagram of reference frames](image)

**Fig. 1.** Definition of the grain local frame \{^g\mathcal{R}\} and the material frame \{^o\mathcal{R}\} with respect to the global fixed reference frame \{\mathcal{R}\}

Macroscopic quantities, associated with a representative polycrystal or the ice-sheet, are overlined. The superscripts \(^g\) and \(^o\) denote any non-scalar
quantity expressed in the local \( \{ u \} \) and the material \( \{ R \} \) frames respectively. Otherwise, \textit{i.e.} if no superscript is applied the quantity is expressed in the global frame \( \{ R \} \).

Following [4,5], the main assumptions made are as follows:

1. The representative polycrystal experiences a uniform state of stress, \textit{i.e.} the stress in each grain equals the bulk stress applied on the boundary of the polycrystal (Reuss assumption leading to the so-called static or uniform-stress model). Since the ice is incompressible, the following relation holds

   \begin{equation}
   S = \bar{S},
   \end{equation}

   where \( S \) and \( \bar{S} \) are the microscopic and macroscopic deviatoric stresses, defined as functions of the stresses \( \sigma \) and \( \bar{\sigma} \) and the isotropic pressures \( p \) and \( \bar{p} \) by \( S_{ij} = \sigma_{ij} - p\delta_{ij}, \quad \bar{S}_{ij} = \bar{\sigma}_{ij} - \bar{p}\delta_{ij} \), respectively.

2. Each grain exhibits a linear behaviour and is considered as a transversely isotropic continuous medium, whose symmetry axis is the crystal \( c \)-axis.

   As a consequence, the orientation of a grain relative to the global frame \( \{ R \} \) is determined solely by the direction of its \( c \)-axis and is defined by two angles, the co-latitude \( \theta \) and the latitude \( \varphi \) (see Fig. 1).

3. Each grain in the polycrystal occupies the same volume.

4. The total number of grains does not change during the deformation, \textit{i.e.} recrystallization processes, such as grain growth or polygonization, are not taken into account.

### 2.2 Grain behaviour

Following [6] and [4], the ice crystal is assumed to behave as a linear transversely isotropic medium. The simplest relation between the strain-rate \( D \) and the deviatoric stress \( S \) is then expressed by

\begin{equation}
D = \frac{\psi}{2} (\beta S + (1 - \beta)(M_3 S + S M_3 - 2 \text{tr}(M_3 S) M_3)).
\end{equation}

where \( M_3 \) is the structure tensor defined by \( M_3 = c \otimes c \) (\( c \) is the grain \( c \)-axis unit vector : \( c = (0, 0, 1) \) in the local frame). The parameter \( \psi \) is the fluidity, inverse of viscosity, for shear parallel to the basal plane of the grain, and \( \beta \) is the ratio of the shear fluidity in the basal plane to that parallel to the basal plane. \( \beta \) is a measure of the grain anisotropy: when \( \beta = 0 \) the grain can deform only by basal glide, as assumed in many models [7,8,3,9], while \( \beta = 1 \) corresponds to an isotropic grain. Since the ice single crystal deforms mainly by shear parallel to its basal plane [1], the value of \( \beta \) should be significantly less than 1.
2.3 Fabric evolution

Following [6], the fabric of the ice polycrystal is described by an Orientation Distribution Function (ODF), which gives the relative density of grains whose $c$-axes have the orientation $(\theta, \varphi)$ in the global frame $\{R\}$. By definition,

$$
\frac{1}{2\pi} \int_0^{2\pi} \int_0^{\pi/2} f(\theta, \varphi) \sin \theta \, d\theta \, d\varphi = 1. \tag{3}
$$

The expressions of the rate of rotation (or spin) of a grain, $\mathbf{W}$ in the global reference frame $\{R\}$, and $\mathbf{wW}$ in the local frame $\{\mathbf{wR}\}$, are related by

$$(\mathbf{R}^T \dot{\mathbf{R}} + \mathbf{wW} - \mathbf{R}^T \mathbf{W} \mathbf{R})^\mathbf{w} = 0, \tag{4}
$$

where $\mathbf{R}$ is the rotation matrix to pass from $\{\mathbf{wR}\}$ to $\{R\}$, and $\mathbf{w}$ is the $c$-axis unit vector expressed in $\{\mathbf{wR}\}$ ([6]). Since the basal planes of a grain remain parallel to each other during the deformation, the component of the velocity along the $x_3$ axis ($c$-axis), when expressed in the rotating frame of the grain $\{\mathbf{wR}\}$, is a function of $x_3$ only. This leads to the kinematic relations

$$\mathbf{wW}_{13} = \mathbf{wD}_{13}, \quad \mathbf{wW}_{23} = \mathbf{wD}_{23}. \tag{5}
$$

To obtain the rate of rotation of the grain reference frame (i.e. the lattice spin) from (4), it is necessary to make an additional assumption that the spin of the grain $\mathbf{W}$, expressed in the global reference frame $\{R\}$, is equal to the macroscopic spin of the polycrystal (Taylor-type assumption),

$$\mathbf{W} = \mathbf{W}. \tag{6}
$$

With relations (5) and (6), equation (4) provides two relations for the change in grain orientation

$$
\dot{\theta} = -\mathbf{wD}_{13} + \mathbf{W}_{13} \cos \varphi + \mathbf{W}_{23} \sin \varphi, \tag{7a}
$$

$$
\dot{\varphi} \sin \theta = -\mathbf{wD}_{23} - \mathbf{W}_{12} \sin \theta + (\mathbf{W}_{23} \cos \varphi - \mathbf{W}_{13} \sin \varphi) \cos \theta. \tag{7b}
$$

Since recrystallization is not taken into account, the change in the relative number of grains in an interval $(d\theta, d\varphi)$ around $(\theta, \varphi)$ is solely due to grains entering or leaving this interval (there is no spontaneous nucleation or disappearance of grains). As a consequence, at each material point $\mathbf{x}$ in the ice-sheet, the polycrystal fabric is described by the ODF which varies according to

$$
\frac{\partial f \sin \theta}{\partial t} + \frac{\partial f \sin \theta}{\partial x_i} \dot{u}_i + \frac{\partial f \sin \theta}{\partial \theta} \dot{\theta} + \frac{\partial f \sin \theta}{\partial \varphi} \dot{\varphi} = 0, \tag{8}
$$

where $\dot{u}_i$ are the velocity components at $\mathbf{x}$. Fabric evolution is described by equations (7a), (7b) and (8).
Following [6,5], we adopt a parameterized form of the ODF derived from analytical results obtained in [4] for the case of an orthotropic polycrystal with fixed principal directions of loading. This ODF depends on four parameters

\[ f(\theta, \varphi, \tilde{k}_1, \tilde{k}_2, \tilde{k}_3, \tilde{\varphi}) = \left[ \sin^2 \theta \left( \tilde{k}_1^2 \cos^2(\varphi - \tilde{\varphi}) + \tilde{k}_2^2 \sin^2(\varphi - \tilde{\varphi}) \right) + \tilde{k}_3^2 \cos^2 \theta \right]^{-3/2}. \]  

This function describes an orthotropic fabric, with planes of symmetry \((\sigma x_1, \sigma x_2), (\sigma x_2, \sigma x_3), (\sigma x_3, \sigma x_1)\). In the following, the plane \((\sigma x_1, \sigma x_2)\) coincides with the flow plane \((x_1, x_2)\) of the ice-sheet. Since the conservation equation (3) implies that \(k_1k_2k_3 = 1\), only three parameters in (9) are independent. Each parameter \(\tilde{k}_i\) gives the strength of concentration of \(c\)-axes in the direction \(\sigma x_i\) of the material frame \(\{\sigma R\}\) (a small value of \(\tilde{k}_i\) corresponds to \(c\)-axes gathered along direction \(\sigma x_i\); when \(\tilde{k}_i = \tilde{k}_j\) the plane \((\sigma x_i, \sigma x_j)\) is a plane of isotropy). \(\tilde{\varphi}\) is the angle which defines the rotation of \(\{\sigma R\}\) with respect to the global frame \(\{R\}\) in plane \((x_1, x_2)\) (see Fig. 1).

### 2.4 Viscous Law

The macroscopic strain-rate \(\dot{D}\) of the polycrystal, in response to the prescribed deviatoric stress \(\tilde{S}\), is defined as the weighted average of the strain-rates \(D\) of its constituent grains as

\[ \dot{D}_{ij} = \left< D_{ij} \right> = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\pi/2} D_{ij}(\theta, \varphi) f(\theta, \varphi) \sin \theta \, d\theta \, d\varphi. \]  

By substituting (2) into (10), and using expression (9) for the ODF, the linear orthotropic viscous law giving the macroscopic strain-rate as a function of the deviatoric stress is obtained as

\[ \dot{D} = \sum_{r=1}^{3} \left[ \hat{\alpha}_r \operatorname{tr}(\sigma^0 M_r \tilde{S}) \sigma^0 M^D_r + \hat{\alpha}_{r+3}(\sigma^0 \tilde{M}_r + \sigma^0 M_r \tilde{S})^D \right], \]  

where \(\sigma^0 M_r = \psi_r \otimes \psi_r\) are three structure tensors defined by the unit vectors \(\psi_r\) of the material frame \(\{\sigma R\}\) and \((\cdot)^D\) denotes the deviatoric part of \((\cdot)\). The six response coefficients \(\hat{\alpha}_r\), functions of the grain rheological parameters \(\psi\) and \(\beta\), are obtained as follows:

\[
\begin{bmatrix}
\hat{\alpha}_1 \\
\hat{\alpha}_2 \\
\hat{\alpha}_3 \\
\hat{\alpha}_4 \\
\hat{\alpha}_5 \\
\hat{\alpha}_6 \\
\end{bmatrix} = \frac{\psi(\beta - 1)}{2} \begin{bmatrix}
0 & -6 & 12 & 8 & -22 & 8 \\
0 & 6 & -12 & -6 & 6 & 8 \\
2 & -10 & 0 & 8 & 6 & -6 \\
\beta/(2(\beta - 11)) & 1 & -3 & -2 & 6 & -2 \\
\beta/(2(\beta - 11)) & -2 & 3 & 2 & -2 & -2 \\
(2-\beta)/(2(\beta - 11)) & 3 & 0 & -2 & -2 & 2 \\
\end{bmatrix} \begin{bmatrix}
1 \\
J_{50} \\
J_{52} \\
J_{50} \\
J_{52} \\
J_{54} \\
\end{bmatrix},
\]  

(12)
where $J_{30}$, $J_{32}$, $J_{50}$, $J_{52}$ and $J_{54}$ are five moments given by

$$J_{pq} = \frac{2}{\pi} \int_0^{\pi/2} \int_0^{\pi/2} f(\theta, \varphi + \theta) \sin^{2p} \theta \sin^{2q} \varphi \, d\theta \, d\varphi.$$  \hspace{1cm} (13)

For isotropic ice, $f(\theta, \varphi) = 1$, $\bar{k}_1 = \bar{k}_2 = \bar{k}_3 = 1$, the $J_{pq}$ values are $J_{30} = 2/3$, $J_{32} = 1/3$, $J_{50} = 8/15$, $J_{52} = 4/15$ and $J_{54} = 1/5$. Then, the macroscopic orthotropic law (11) reduces to a linearly isotropic viscous law (i.e. Glen’s law with $n = 1$)

$$\mathbf{D} = \frac{\bar{B}_1}{2} \mathbf{S}, \hspace{2cm} (14)$$

where the fluidity $\bar{B}_1$ is related to the grain rheological parameters by [4,5]

$$\bar{B}_1 = \frac{\psi}{5}(3\beta + 2). \hspace{2cm} (15)$$

It follows from (15) that the ratio of the fluidity $\psi$ for shear parallel to the grain basal plane to the fluidity $\bar{B}_1$ of isotropic ice reaches its maximum value of 2.5 when the grain behaviour is the most anisotropic one (i.e. when $\beta = 0$). This value is lower than the experimental value of 10 obtained by [10]. Therefore the influence of anisotropy on ice-sheet flow, as given by the present model, is under-estimated.

3 Ice-sheet flow model

The stationary flow of an ice-sheet with a fixed geometry (the vertical at the dome is a symmetry axis) is solved for given boundary conditions (no stress and isotropic fabric at the free surface, no-sliding on the bed-rock).

3.1 Velocity field

Following [11], the solution in terms of the velocity $\vec{u}$ and isotropic pressure $\bar{p}$ is obtained, for a given fabric field, by minimizing, among the admissible velocity fields, the functional

$$J(\vec{u}, \bar{p}) = \int_D \left( \bar{\phi}_D + \bar{p} \bar{D}_{ii} - \rho g \vec{u}_z \right) \, d\mathbf{D} - \int_E \bar{\sigma}^{\phi} \vec{u}_i \, n_i \, ds, \hspace{2cm} (16)$$

where $\bar{\sigma}^{\phi}$ is the atmospheric pressure applied on the free surface $E$ and $\bar{\phi}_D$ is the dissipation potential of the orthotropic polycrystal which gives the deviatoric stress as a function of the strain-rate as $\mathbf{S} = \partial \bar{\phi}_D / \partial \mathbf{D}$. This relation can be put into the same form as (11) by interchanging $\mathbf{D}$ and $\mathbf{S}$, and replacing the fluidities $\bar{\alpha}_i$ with six viscosities $\bar{\eta}_i$ whose expressions are obtained by identification (see [12] for these relations).
The functional equation (16) is solved by the finite-element method, using six-node triangular elements with a quadratic interpolation of the velocities and a linear interpolation of the pressure. The ODF (9) is obtained by a quadratic interpolation of the parameters $\tilde{k}_1$, $\tilde{k}_2$ and $\psi$ given at each node of the mesh.

3.2 Fabric field

Following [13], the ODF parameters corresponding to stationary flow are obtained by solving equation (8) along the flow streamlines computed from the finite-element solution for the velocities and assuming that the ice is isotropic at the ice-sheet surface. Denoting by $s$ the curvilinear co-ordinate along a streamline and $\tilde{u}_s$ the velocity tangent to this streamline, under the assumption of stationarity $\partial(f \sin \theta)/\partial t = 0$, equation (8) transforms into

$$\frac{\partial f \sin \theta}{\partial s} - \tilde{u}_s + \frac{\partial f \sin \theta}{\partial \theta} + \frac{\partial \tilde{u}_s}{\partial \varphi} = 0.$$  \hspace{1cm} (17)

The ODF parameters at each node $M$ of the finite-element mesh are computed in two stages as follows:

- the streamline passing through $M$ is computed from $M$ to the ice-sheet surface by solving the set of equations $dx_i/dt = -\tilde{u}_i$ (upstream procedure). During this phase, $\tilde{S}_{11}, \tilde{S}_{22}, \tilde{S}_{12}$ and the rotation rate $\tilde{W}_{12}$ are calculated at each time step by using the finite-element solution for the velocity and the constitutive law (11), then stored.

- the evolution of the ODF parameters along the streamline is calculated from the surface where ice is assumed to be isotropic. By assuming that the directions of the principal stresses do not change significantly during the time step, the ODF given by (9) satisfies (17) and the parameters $\tilde{k}_i$ are shown (see [12]) to be the solution of

$$\frac{d\tilde{k}_1}{\tilde{k}_1} = (\psi/4)(\tilde{S}_{11} + \tilde{S}_{22} + \tilde{S})dt, \quad \frac{d\tilde{k}_2}{\tilde{k}_2} = (\psi/4)(\tilde{S}_{11} - \tilde{S}_{22} - \tilde{S})dt,  \hspace{1cm} (18)$$

where $\tilde{S}^2 = (\tilde{S}_{11} - \tilde{S}_{22})^2 + 4\tilde{S}_{12}^2$. During a time step $dt$, the orientation change of the orthotropic frame $\{\psi R\}$, i.e. $d\psi$, is taken as the weighted average of the rotation $d\tilde{W}$ of the grains given by (7b). These equations are solved by using the Runge-Kutta method (with initial condition at the surface $\tilde{k}_1 = \tilde{k}_2 = 1$).

3.3 Coupled problem

The velocity and fabric fields corresponding to stationary flow are calculated by solving iteratively the velocity problem (equation (16) for a given fabric field, then the fabric problem (equations (18)) for a given velocity field, until
convergence is achieved (i.e. when the norm of the relative deviation of each variable \(a, \tilde{k}, \tilde{\psi}\) for two consecutive iterations is less than \(10^{-3}\)). It has been verified that the choice of the initial fabric field (respecting the condition of isotropy at the surface) does not affect the results.

4 Applications

4.1 Conditions of the numerical simulations

All the variables are made dimensionless by using the depth at the dome \(H_0\), the gravity forces \(\rho g\) and the fluidity of isotropic ice \(\tilde{B}_1\) at \(-10^\circ\)C as scaling values (stresses are scaled by \(\rho g H_0\), and velocities by \(\tilde{B}_1 \rho g H_0^2\)). Dimensionless variables are denoted by a tilde.

We adopt a simplified ice-sheet geometry, with a flat bed-rock and a fixed surface elevation given by Vialov's profile [14] \(\tilde{H}^4 = 1 - (\tilde{c} \tilde{r})^2\), where \(c = H_0/L\) is the aspect ratio of the ice-sheet, fixed to the value 0.018 in the following. With this fixed geometry, the accumulation-rate corresponding to stationary flow must be considered as a variable determined by the solution of the flow problem.

Since the fluidities \(\tilde{B}_1\) and \(\psi\) verify relation (15), the dimensionless variables of the model, including the ODF parameters, depend only on the grain anisotropy parameter \(\beta\) and the temperature. In the following the results of three simulations, whose conditions are summarized in Table 1, are compared. The non-isothermal temperature field named "GRIP" is deduced from the profile shown on Fig. 2a measured in the GRIP borehole, [15], by assuming that the temperature is a function of the reduced elevation \(\tilde{x}_2 = x_2/H\) only. Note that since the viscosity of ice exhibits an exponential dependence on the temperature, changing the reference temperature would affect the velocities by a scaling factor (as a change in the value of \(\tilde{B}_1\) would do) but not the ODF as can be inferred from (17).

<table>
<thead>
<tr>
<th>Test</th>
<th>(\beta)</th>
<th>Temperature profile</th>
</tr>
</thead>
<tbody>
<tr>
<td>Test 1</td>
<td>0.25</td>
<td>&quot;GRIP&quot;</td>
</tr>
<tr>
<td>Test 2</td>
<td>0.25</td>
<td>isothermal</td>
</tr>
<tr>
<td>Test 3</td>
<td>0.1</td>
<td>&quot;GRIP&quot;</td>
</tr>
</tbody>
</table>
Fig. 2. (a) Evolution of the temperature as a function of the reduced elevation $\tilde{x}_2$ as measured by [15] in the GRIP ice core (solid line) and the isothermal profile (dashed line). (b) Horizontal velocity as a function of the reduced elevation $\tilde{x}_2$ at $\tilde{x}_1 = 10$ for Test 1 (solid line), Test 2 (dashed line) and Test 3 (dotted line).

4.2 Influence of the Temperature

The influence of the temperature is assessed by comparing Test 1 and Test 2. The constant temperature in Test 2 was chosen so that Test 1 and Test 2 give the same surface horizontal velocity at $\tilde{x}_1 = 10$.

Owing to the increase in the temperature at GRIP by 23.3°C from the surface to the bedrock, the fluidities $B_1(T)$ and $\psi$ are increased by a factor of approximately 30 from the top to the bottom. On the other hand, due to the development of anisotropy from the surface to the bedrock, the shear fluidity (in the flow plane) can be increased by a maximum factor of about 1.8 (obtained by assuming that all the $c-$axes are aligned along the vertical, so that the macroscopic shear fluidity equals the grain fluidity for shear parallel to the basal plane $\psi$ given by (15) with $\beta = 0.25$). However, Fig. 3 shows that the "GRIP" temperature field (Test 1) significantly slows down the concentration of the fabric (given by the smallest parameter $k_2$) as a function of depth, compared to the isothermal flow: the difference in $k_2$ between Test 1 and Test 2 increases with depth until it reaches the threshold value of $k_{2c} = 10^{-3}$ for which it is assumed that all the grains have the same orientation and the fabric cannot concentrate any further ($k_{2c}$ is reached at $\tilde{x}_2 = 0.4$ in Test 2 and at $\tilde{x}_2 = 0.2$ in Test 1). The two other parameters $\tilde{k}_1$ and $\tilde{k}_3$ satisfy $\tilde{k}_1/\tilde{k}_2c > 10^6$ and $\tilde{k}_3/\tilde{k}_2c > 100$ when the threshold value $k_{2c}$ is reached.
Fig. 3. Evolution of fabric parameters $\hat{k}_2$ (a) and $\hat{\varphi}$ (b) as a function of the reduced elevation $\hat{x}$ at $\hat{x}_1 = 10$ for Test 1 (solid line), Test 2 (dashed line) and Test 3 (dotted line).

Even if the flows are very similar (owing to the adjustment of the temperature in Test 2), horizontal velocity profiles (see Fig. 2b) show that for Test 2 the deformation is relatively homogeneous while the effect of the GRIP temperature field is to concentrate the deformation in a basal layer (about 20% of the total thickness). Ice being less deformed near the surface, fabric concentration is slower than for isothermal conditions. An additional consequence is that the direction of the preferred orientation of the grains is closer to the vertical for Test 1 than for Test 2 (see $\varphi$ in Fig. 3b). Also, owing to the shape of the velocity profiles, the accumulation rate for the "GRIP" flow is 1.2 to 1.4 greater than for the isothermal.

4.3 Influence of grain anisotropy

The influence of the grain anisotropy, described by $\beta$, is shown by comparing Test 1 and Test 3 (see Table 1). At the scale of the polycrystal, the rate of fabric concentration increases as the anisotropy of the grain increases (i.e. $\beta$ decreases) [6,4,5]. This holds at the scale of the ice-sheet since the decrease of $k_2$ with depth is faster for Test 3 ($\beta = 0.1$) than for Test 1 ($\beta = 0.25$) as shown in Fig. 3.

Since for a given fabric the macroscopic fluidity increases with the grain anisotropy (see relations (11), (12), (15) and [12]), the effect of the rate of fabric concentration on the velocities is enhanced. Accordingly, the accumulation rate for Test 3 is about 1.3 times larger than that of Test 1.
5 Conclusion

A model for the flow of an ice-sheet exhibiting strain-induced anisotropy has been presented. The fabric of an orthotropic ice polycrystal is described by using an ODF which depends on three independent parameters only. Assuming a linear transversely isotropic behaviour of the grain and a uniform state of stress, the constitutive law of the polycrystalline ice is obtained by homogenization. As a consequence of the assumed linear behaviour of the grain, the polycrystal behaviour is also linear. This assumption seems to be well adapted to describe polar ice rheology from the surface to the two-thirds of the ice sheet thickness [10]. Deeper, as the temperature and the stress increase, the value $n = 3$ would be more suitable.

The model has been applied to the simulation of the stationary flow of a two dimensional ice-sheet with a fixed geometry. The finite-element flow computation was carried out by assigning the values of the ODF parameters at each node of the mesh, then the evolution of the fabric was calculated along the flow streamlines derived from the velocity field, and the process was repeated until convergence was achieved. For a given aspect ratio of the ice-sheet, the influence of both temperature and grain anisotropy on the velocity and fabric fields has been shown. A field of temperature derived from GRIP field measurements slows down the rate of fabric evolution with depth (compared to that obtained for an isothermal flow) and an increase of the grain anisotropy leads to an increase of both the velocity and the rate of fabric evolution.

This study demonstrates the efficiency of our model to simulate the flow of ice-sheets exhibiting strain-induced anisotropy and shows that it could be used to determine the velocities and the ice fabrics for more complex bedrock and surface topographies, for instance in the vicinity of the drilling sites in Antarctica and Greenland.

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